Determination of minimum factor of safety using a genetic algorithm and limit equilibrium analysis

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ABSTRACT
This paper presents the continuous parameter genetic algorithm combined with the rigid finite element method to identify minimum factors of safety for slopes. Whereas for homogeneous slopes, Bishop's and RFEM analyses lead to similar predictions for global minimum factor of safety and location of critical slip surface, it is shown that predictions differ for layered systems. It is important that layer rigidity is included for layered systems.

RÉSUMÉ
Cet article présente le paramètre continue de l'algorithme génétique combiné avec la méthode des éléments finis rigides pour identifier les facteurs minimales de sécurité des pentes. Alors que pour les pentes homogènes, la méthode de Bishop et celle des Éléments Finis Rigides conduisent à des prédictions similaires pour le facteur de sécurité globale et de la position de la surface de glissement critique; il est montré que les prédictions sont différentes pour un système hétérogène avec plusieurs couches. C'est important que la rigidité des couches soit prise en compte pour un système hétérogène à plusieurs couches.

1 INTRODUCTION

With the availability of numerical tools such as the finite element method, the geotechnical engineer is capable of realistically solving complex problems. Owing to the uncertainty surrounding the initial conditions, boundary conditions and distribution, as well as history-dependent properties, the quality of the predictions are most often undermined. Simple methods often provide solutions that are adequate for practice given that the engineer is often interested in answering only two questions: (1) Is the structure stable?; and (2) Assuming a sufficient factor of safety, what movements are to be expected? These questions may be answered by uncoupling the failure and deformation analyses.

Limit equilibrium analysis presents a long-standing simple framework for slope stability analysis. Many methods exist and the selection of the "best" may not be a trivial task. For a detailed discussion of the relative merits of various methods, the reader is referred to Fredlund & Krahn (1977). Implicit in the methodology is the use of a search algorithm to identify the surface that minimizes the factor of safety. The presence of many local minima can easily mislead traditional root finding methods. To help overcome this problem, various algorithms have been developed, such as: "grid and radius" methods (Chen and Shao 1983); "Monte Carlo" random search technique (Malkawi et al. 2001); and "genetic algorithm" search procedures (McCombie & Wilkinson 2002; Sengupta & Upadhyay 2005; Zolfaghari et. al., 2005).

The objective of this paper is to present a continuous parameter genetic algorithm with emphasis on combining it with the rigid finite element method (RFEM) to identify minimum factors of safety of slopes. We begin by providing a short overview of the RFEM and then describe a continuous genetic algorithm, followed by examples that compare RFEM predictions with those obtained by Bishop's simplified method.

2 BACKGROUND

2.1 Rigid Finite Element Method

Similar to traditional limit equilibrium analyses a potential failure surface is assumed, a priori. The domain is subdivided into a series of rigid vertical slices as shown in Figure 1 for an assumed circular slip surface that is characterized by the center of the circle (X_c, Y_c) and its radius R.

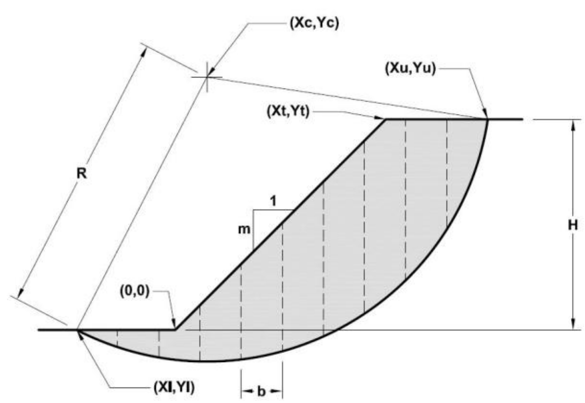

Figure 1. Geometry of slope stability problem

The slices are assumed to be rigid, with relations defined for the interaction of elements; *i.e.*, between inter-slice forces and the relative movement of adjacent slices. Similar relations exist between the slip surface and the rigid base using non-linear springs as shown in Figure 2. The movement of each element is constant and is defined in terms of horizontal and vertical displacement components. Following a finite element methodology, a system of equations is assembled consistent with virtual work for a specific slip surface. Since the interactions of slices are non-linear, an iterative scheme must be used to solve the matrix equation. This procedure satisfies both local and global force equilibrium. Similar to traditional approaches, a comparison is made between resisting and mobilized moments to determine the global factor of safety. An advantage of the RFEM procedure is that relative movements are determined as part of the solution, which allow local factors of safety to be readily determined. The reader is referred to Stolle & Guo (2008) for details.

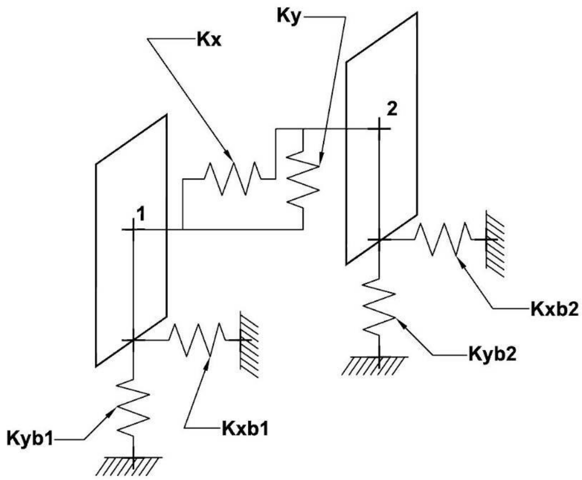

Figure 2. Schematic of inter-slice and basal springs.

2.2 Genetic Algorithm

A "genetic algorithm" begins with a random sample of failure surfaces within a suitable domain, characterized by (X_c, Y_c, R); see Table 1. The selection of the domain is an important step as too small of a search region may exclude the combination that defines the global minimum. On the other hand, too large of a search region may take too long to converge (due to the larger number of possible combinations) or lead to an unreasonable slip surface depending on the type of analysis used.

Using the principle of natural selection, the critical factor of safety is determined by systematically adjusting the failure surface parameters of previously generated results. At the same time, it allows for an appropriate level of random searching in order to prevent converging too quickly on local minima. This approach strikes a balance between the systematic, resource intensive "grid and radius" method and the highly randomized "Monte Carlo" method.

Table 1. Parameter ranges used for genetic algorithm

Parameter[1]	Minimum	Maximum
X_c	-0.25H	H/m + 0.25H
Y_c	1.25H	2.5H
R	0.2H	2.5H

[1] See Figure 1: *H* = height of slope, *m* = grade of slope (rise/run).

The binary number "genetic algorithm" search method has received the most attention; see previous references. This paper describes an alternative genetic algorithm presented by Haupt and Haupt (1998) that simply stores the combination of parameters as a vector in decimal (continuous) form rather than in binary form. The advantages of this approach include: manipulation of the parameters in crossover and mutation routines are more transparent and intuitive; and less computation time is spent on bookkeeping to convert the parameters back and forth between binary and decimal notation.

The general algorithm involves stages: initialization, evaluation, sorting, reproduction (crossover) and mutation. One performs the last three steps iteratively for a specified number of generations. The processes of crossover and mutation allow the model to identify the global minimum even while converging on a local minimum through random mutations in the parameters. It is important to use a set number of iterations as a stopping criterion, since a relative-error-based criterion may cause the algorithm to complete prematurely at a local minimum.

2.2.1 Initial Population and Evaluation

After the search domain is defined, the first step in the genetic algorithm is to generate an initial "population" of candidates for minimum global factor of safety. "Chromosomes" (X_c, Y_c, R) containing randomly selected values for each of the parameters within their respective ranges compose this initial population. The size of the initial population is an important parameter in the genetic algorithm. The factor of safety F_s is then evaluated for each chromosome using RFEM.

2.2.2 Sorting

Once the factors of safety have been evaluated for the current population, they are sorted from lowest to highest global factor of safety. This is in preparation for the reproduction and mutation processes, where only the most optimal configurations carry their "genes" through to the next generation.

2.2.3 Reproduction (Crossover)

It is through crossover that the genetic algorithm is able to converge on a particular value through the process of natural selection. The reproduction process begins by selecting a specified upper portion of the population (typically 50%) to enter the mating pool. This represents "survival of the fittest" as only the most optimal proportion of the population is able to pass traits on to the next generation. The most optimal solution(s) (typically ~10-15%, but rounded up to the nearest integer) are copied directly into the next generation. This means that the search routine always keeps the most optimal solution so that the minimum global factor of safety never increases. This is akin to the fittest members of the population having greater longevity and surviving for an extra "mating season" (in addition to passing their genes on to the next generation).

The algorithm then assigns appropriate weights to the members of the mating pool according to the corresponding global factor of safety by the following operations:

1. Subtract the minimum factor of safety of the chromosomes not selected for reproduction from the factors of safety of all chromosomes selected for reproduction. This reverses the weighting scheme, as this is a minimization problem.
2. Divide each of the (reversed) factors of safety in the mating pool by the sum of the (reversed) factors of safety to obtain the individual weight of each.
3. Add the sum of the weights for lower factors of safety to obtain the cumulative weight for each individual chromosome.

Next, the algorithm selects pairs of parent configurations in order to fill the remaining slots in the next generation's population. If there are an odd number of slots remaining, the program moves the next most optimal solution after those already deemed "fittest" directly into the next generation. It then carries out the selection process by generating random numbers between 0 and 1 and selects the first parent in the mating pool that has a cumulative weight greater than or equal to this value. This selection process, combined with the weighting scheme described previously, ensures that chromosomes with significantly lower factors of safety will have a greater relative probability of being selected, while a population consisting of chromosomes with nearly equal factors of safety will have a uniform probability of being selected

The next step is to carry out the process of crossover in order to fill the population for the next generation, which begins by generating a random number between 0 and 1. If this number is less than a specified probability (typically 70~90%), crossover occurs. Otherwise, the program copies the two parents directly into the next generation. If crossover is to occur, the program performs a form of interpolation on one parameter and crossover on the remaining parameters. Figure 3 shows a schematic of the process. The figure uses the *Y* term as an example. Since the number β is a randomly generated between 0 and 1, the maximum change that the process applies to the parameters is the difference between them. Thus, if the parameters are similar then two similar offspring will be generated and vice versa if the two parameters are far apart. It should be noted that there are a number of ways in which crossover can occur, besides what is shown in Figure 3. If *X* were selected for interpolation, either *Y* or *R* would have been randomly selected to be crossed; if *R* were interpolated, *X* or *Y* would have been crossed; thirdly, as in Figure 3, if *Y* is interpolated, either *X* or *R* is crossed. This removes bias toward crossing one parameter or the other in all cases. Finally, the program places the two offspring into the population for the next generation. This process repeats until the population for the next generation is filled.

$$parent_1 = \left[X_m \quad Y_m \quad R_m \right]$$
$$parent_2 = \left[X_d \quad Y_d \quad R_d \right]$$
$$\ldots$$
$$Y_{new1} = Y_m - \beta(Y_m - Y_d)$$
$$Y_{new2} = Y_d + \beta(Y_m - Y_d)$$
$$\ldots$$
$$offspring_1 = \left[X_m \quad Y_{new1} \quad R_d \right]$$
$$offspring_2 = \left[X_d \quad Y_{new2} \quad R_m \right]$$

Figure 3. Determination of chromosomes via crossover.

2.2.4 Mutation

The process of mutation allows the genetic algorithm to survey the search region for alternative minima. It prevents the algorithm from converging too quickly on a local minimum and allows the algorithm an exit from the region of a local minimum if it finds a lower value; i.e., the global minimum. If a lower minimum exists, it may take a number of generations for the algorithm to find it; in fact, the closer the current minimum is to the global minimum without being in the region of the global minimum, the less probable it is to locate the global minimum. This is the main reason why setting the number of generations is preferable to a convergence criterion for the genetic algorithm. The number of mutations on a given generation is $n_{mut} = 3 \times p_{mut} \times (n_{pop} - n_{fit})$ with n_{mut} = total number of mutations; p_{mut} = probability of mutation; n_{pop} = number of chromosomes in population; and n_{fit} = "fittest" proportion of population. Table 2 lists values used here.

Table 2. Parameters used for optimization

Population (n_{pop})	15
Fittest Population ($p_{fit} = n_{fit} / n_{pop}$)	0.10
Mating Pool Proportion ($p_{mat} = n_{mat} / n_{pop}$)	0.50
Crossover Probability (p_{cross})	0.90
Mutation Probability (p_{mut})	0.25
Number of Generations (n_{gen})	100

Following the crossover process of each generation, a vector of (chromosome, parameter) addresses are selected (excluding the "fittest" proportion). For each location selected for mutation, the program selects a random value within that parameter's corresponding range to replace the current value.

3 NUMERICAL EXAMPLES

This study considered various scenarios. The essence of the genetic algorithm together with the RFEM (and its advantages) may be illustrated by considering the results of two particular cases. The parameters *E*, *v*, *φ* and *c* denote elastic modulus, Poisson's ratio, friction angle and cohesion, respectively.

3.1 Case 1 – Layered Embankment

Case 1 is an embankment consisting of three strata, plus a bottom layer for bedrock. The upper and lower strata are identical layers of sand with φ = 40°, E = 1.0x10^5 kPa, and v = 0.33. The middle stratum is a layer of mixed soil with φ = 10°, c = 30 kPa, E = 5x10^4 kPa, and v = 0.45. The geometry of the slope is defined by m = 5/6, H = 25 m (see Figure 1), layer thicknesses of 12.5 m for the sand and 5 m for the mixed soil and γ = 18.5 kN/m^3 for all soils. The bedrock lies 5 m below the base of the slope. Minimum factors of safety using RFEM and Bishop's method together with the genetic algorithm were obtained for several runs, each one being a full analysis.

Table 3 summarizes the statistics from the runs, with Figure 4 presenting the minimum F_s. Originally 5 runs were carried out to confirm that the algorithm would converge to a consistent minimum F_s. One observes from the figure that F_s is relatively constant when using Bishop's method for slope stability, but the RFEM predictions show sensitivity to the (random) seed value. Additional runs were therefore completed to further investigate sensitivities. The sensitivity that was originally noted was confirmed. More important is the observation that the minimum value was achievable only after 65 runs. In other words, there is no guarantee that a genetic algorithm will converge quickly. It depends on the sensitivity of the evaluation function to input. Table 3 clearly shows that there is considerable variation in critical failure surface compared to changes in minimum F_s for this particular problem. The RFEM stability procedure predicted a lower global minimum.

Table 3. Case 1: Genetic algorithm statistics for 85 runs.

	X_c (m)	Y_c (m)	R (m)	F_s
	Bishop			
Average	10.97	44.38	39.55	1.028
SD	3.83	6.81	7.05	0.010
Min.	1.25	31.36	26.26	1.011
Max.	16.44	58.21	53.62	1.051
	RFEM			
Average	10.18	44.61	39.47	0.95
SD	4.56	7.68	8.06	0.10
Min.	0.36	25.51	23.54	0.67
Max.	17.16	63.48	58.98	1.10

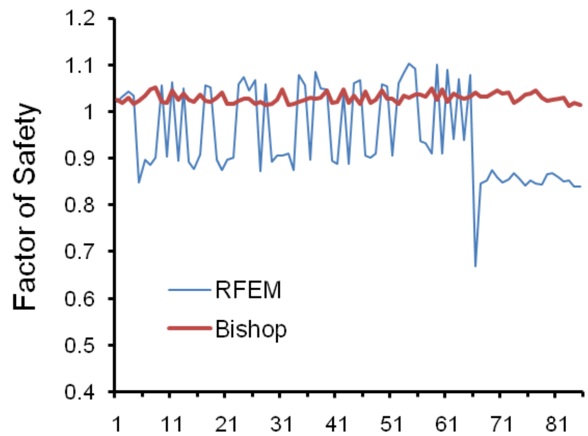

Figure 4. Minimum factor of safety as function of run.

The critical surface for run 5 (minimum $F_s \approx$ 0.84) is shown in Figure 5, along with the variation of local F_s along the failure surface as determined from the sliding law; see Stolle and Guo (2008). An examination of the local factors of safety along the slip surface reveals that F_s is highest near the base of the failure surface and steadily decreases as one moves up the surface. This is expected as the level of confinement also decreases as one moves up the surface. Although the mixed soil is the key for the failure mode given that the F_s is slightly greater than 1 in the sand, the transfer in load due to the instability of the mixed soil causes the local factors of safety to further decrease in the sand when load is redistributed. Bishop's procedure was not capable of transferring load and therefore the search for the minimum tended to focus on mechanisms that were restricted to the sand. This simple example demonstrates the importance of accommodating load redistribution in stability analysis.

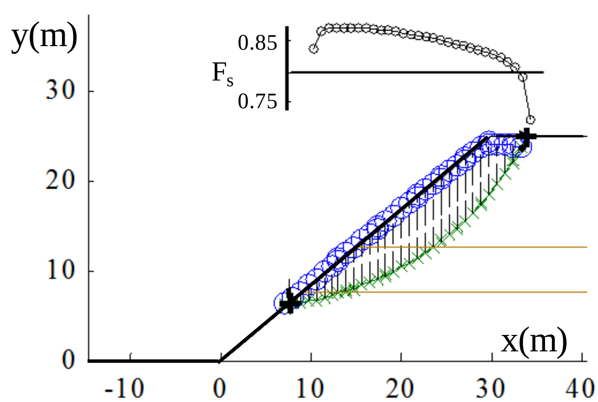

Figure 5. Predicted failure surface (RFEM) for run 5.

The minimum factor of safety corresponding to the RFEM analysis was found to be approximately 0.61.

Figure 6 shows the critical surface and compares it with that predicted when using the Bishop algorithm (solid line for RFEM, dashed line for Bishop's method). The surface in Figure 6 is not much different from that of Figure 5.

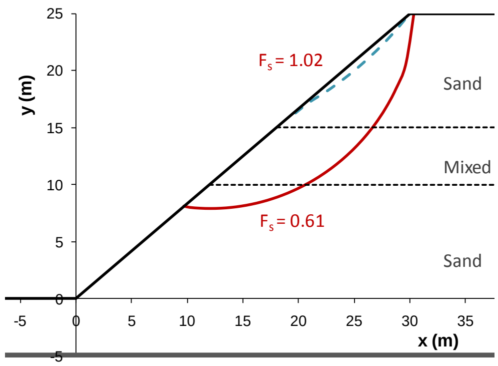

Figure 6. Failure surface for global minimum, Case 1.

3.2 Case 2 – Narrow, Weak Layer

Case 2 corresponds to a slope similar to that used by Fredlund & Krahn (1977) for studying the various classical methods of slope stability analysis. The slope consists of a dense, mixed soil lying over a bedrock base. At the interface of these two layers, there is a very thin layer of weak frictional sand. The upper layer is characterized by γ = 18.85 kN/m^3, φ = 20°, c = 29 kPa, E = 7.5x10^4 kPa, ν = 0.4, and has a thickness of 12.2 m. The thin middle layer is weak frictional sand having γ = 18.85 kN/m^3, φ = 10°, E = 5.0x10^4 kPa, ν = 0.45, and is 0.5 m thick. The slope has a grade m = 0.5 with H = 12 m. Since their paper only considers classical methods of analysis, it does not provide values for elastic modulus or Poisson's ratio, so these values were assumed.

The emphasis with this example is the effect of the number of slices used on the global factor of safety. Figure 7 compares the predictions from the Bishop's and RFEM simulations. It is clear that the minimum factor of safety for each method levels off above ≈ 10 slices. There is a slight increase for Bishop's method, but this may be due to slight increased numerical errors when the slices become thinner. One observes that Bishop's method predicts lower F_s, which may indicate that the method is conservative due to the simplifying assumptions. One must, however, be careful with making conclusions and generalizing as there are there are special cases (such as Case 1) where RFEM predicts lower values as it is capable of accommodating failure modes resulting from stiffness-dependent load transfer, which Bishop's method ignores.

Figure 8 compares the critical surfaces predicted by each stability analysis procedure for the 10 slice analyses. The corresponding local safety factors (RFEM) are also shown. Both predict similar critical surfaces, although the minimum factor of safety for each is different. With the RFEM one gets a better picture of the way the slope moves. While there is a rotation for both procedures, for RFEM, greater relative movement between slices is observed at the crest, which is consistent with the local F_s, being smaller; *i.e.*, large movements being consistent with greater yielding.

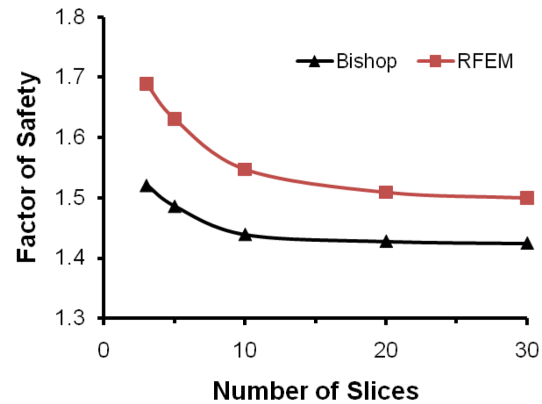

Figure 7. Comparison of factors of safety from Bishop's and RFEM analyses.

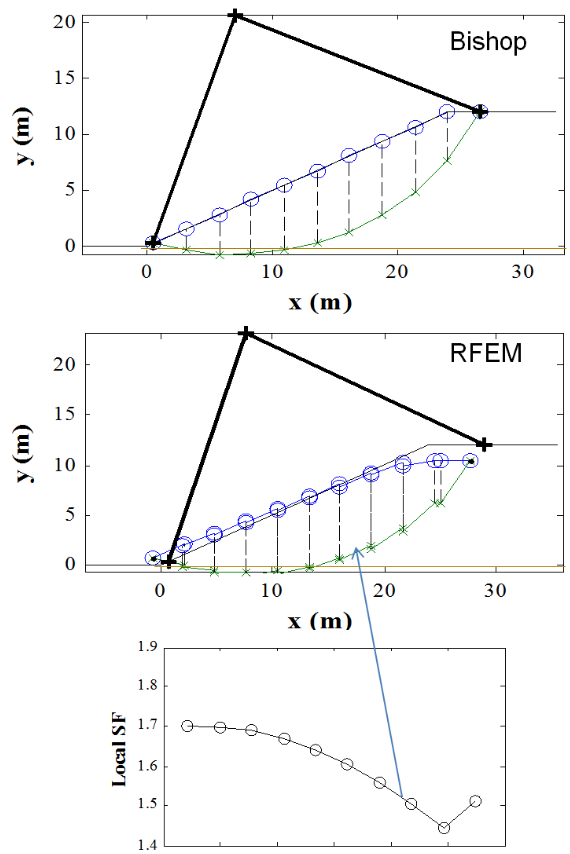

Figure 8. Comparison of failure surfaces from Bishop's and RFEM analyses.

4 CONCLUDING REMARKS

Only a few results have been shown in this paper. Nevertheless, observations presented in this section take into account the experiences from the entire study. This study makes the following conclusions:

- For homogeneous slopes, Bishop's method and RFEM analysis produce comparable results both in terms of the numerical value of the global minimum factor of safety and the location of the critical slip surface.
- For slopes with multiple strata, it is important to consider whether the different layers have significantly different values for the stiffness terms, *E* and *v*. If the values differ significantly, classical approaches such as Bishop's method may not give accurate numerical estimates of the global minimum factor of safety. This is likely due to how the soil mass transmits lateral stresses, inducing shear flow along the interface between two strata.
- For slopes with multiple strata, Bishop's method and RFEM analysis produce comparable results for the location of the critical slip surface. It may be most computationally efficient to use Bishop's method analysis to arrive at the critical slip surface, but finally evaluate the actual factor of safety using RFEM. Nevertheless, converging to a global minimum requires more generations for a multilayer slope.
- For cohesionless and mixed material slopes, the (X_c, Y_c, R) parameters that define the critical surface may vary considerably for a given F_s. This implies that such slopes may be less sensitive to variations in the macrostructure of the slope (e.g. inclusions such as boulders) as there is a wide range of equally critical failure surfaces.
- For slopes of cohesive material with $\varphi = 0°$, the critical surface tends to be more unique, with minimal variation in the (X_c, Y_c, R) parameters. This means that if there is a region of localized strength along the critical surface (*e.g.* a shear key), stability increases significantly. The reverse is true for regions of localized weakness along the critical failure surface.
- The number of slices used in the analysis has an impact on predictions. With too few (< 10), the analysis overestimates the minimum factor of safety as an accurate representation of the failure surface geometry is lacking. On the other hand, too many slices may prove computationally inefficient and introduce an undesirable level of numerical error into the computation. It is therefore optimal to use a balanced number of slices in the range [20, 30] to get an accurate solution while keeping the computational expense in a satisfactory range.

The following was observed when comparing predictions by binary number and continuous parameter algorithms at the beginning of the study:

- The binary number method tends to be more powerful at random searching for a given set of genetic algorithm parameters. This gives the binary number method an advantage in situations where there are many local minima because it is more likely to discover the region of the global minimum. The binary number method is therefore best suited to situations where a minimum number of runs are necessary due to either time or computational constraints since it is more likely to find the global minimum in fewer runs.
- The continuous parameter method, while slightly less effective at conducting a random search of the minimum F_s, it does tend to run faster due to less bookkeeping required in converting to and from binary numbers during each iteration. In most situations, the ability to do more runs in less time is advantageous because having more output provides a better idea of how unique the critical surface is and how much influence it has on overall slope stability.

Although the procedure has been tested on various slopes, additional testing is required to check robustness of the methodology and influence of interlayer stiffness on the minimum factor of safety, as well as fully explore the limitations of the rigid finite element procedure.

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