Discontiuum mechanics of isotropic multilayered soils beneath a uniform vertical load

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ABSTRACT
This paper deals with the basic principles of the deterministic discontinuum mechanics and its application to the simplification of the Burmister´s problem. For this purpose, the soil is modeled as a Bravais isometric lattice that is governed by the mean value principle, which is stated as the identity between a quantity at a homologous point and the mean value of the quantities at the neighboring nodes, located within the polyhedron of influence. Particularly, consistently with the result of the isotropic continuum mechanics, the biharmonic equation is obtained by averaging a stress function of the Kirchhoff´s kind. But the discrete nature of the discontinuous medium allows the simplification of the problem even more, by considering a partial or biased polyhedron. Taking advantage of this property, expressions for displacements, stresses and strains in a multilayer system, subjected to a uniformly distributed vertical load applied over the surface, are derived; which, compared to the Burmister´s results, are sufficiently rigorous, explicit, elementary, simple and accessible to the practical engineer.

RESUMEN
Este artículo trata de los principios básicos de la mecánica determinística del medio discontinuo y su aplicación a la simplificación del problema de Burmister. Con este propósito, el suelo es modelado como una red tridimensional isométrica de Bravais, que es gobernada por el principio del valor medio, el cual se define como la identidad entre una cantidad correspondiente a un punto homólogo y el promedio de las cantidades correspondientes a los nodos vecinos, localizados dentro del poliedro de influencia. Particularmente, en concordancia con el resultado de la mecánica del medio continuo isotrópico, la promediación de una función del esfuerzo del tipo de Kirchhoff conduce a la ecuación biharmónica. Pero la naturaleza discreta del medio discontinuo permite la simplificación del problema aún más, considerando un poliedro parcial o sesgado. Aprovechando esta propiedad, se derivan expresiones para los desplazamientos, esfuerzos y deformaciones en un sistema multicapa sometido a una carga vertical uniformemente distribuida sobre la superficie, las cuales, en comparación con los resultados de Burmister, son suficientemente rigurosas, explícitas, elementales, simples y accesibles al ingeniero práctico.

1 INTRODUCTION

The stress distribution in a discontinuous substance is of greater complexity, because it depends not only on their structure but also on the nature of the contact forces, which can exhibit high hyperstaticity. Historically, Trollope (1956) was the first to calculate the stresses caused by the self-weight of a two-dimensional dense and orderly ensemble, consisting of ellipses in contact, overcoming the static indeterminacy by means of a coefficient of arching. When these equations are written in a compact fashion, it becomes clear that the stress components satisfy the wave equation (Yanqui 1980), being in the middle of the elliptical nature of the theory of elasticity and the higher hyperbolic character of the theory of plasticity. Moreover, the principle of the Trollope´s centroidal reactions has been used to calculate the stresses caused by the self-weight of an N-dimensional jointed rock mass, and to calibrate the strength parameters of granular soils (Yanqui 1982).

Currently, several theories have been developed to describe the stress transmission mechanism in granular media, based on the statistical mechanics (e.g. Aste et al. 2002, Bouchaud 2002), which takes into account the randomness of the grain distribution and the contact forces. Alternatively, in concordance with the crystallographic description of granular packings, a simpler and more applicable theory has been raised, based on the so-called principle of the mean value (Yanqui, 1995), whose target is the stress function rather than the contact force. Furthermore, since this principle can be arbitrarily biased, solutions of different levels of accuracy can be established. The simplified mechanics of a layered discontinuum is an example of application of this methodology.

2 DISCONTINUUM MECHANICS

2.1. The structure of the ideal matter

All natural substances are discontinuous because they are made of particles and voids. In this context, a particle is defined as a unit of matter, such as a block in rock masses, a grain in soils, a mineral in rocks, a molecule in crystals, and so forth. The assemblage of these units is multiscale and multiphase and may be in a nearly ordered pattern or in a random way. But the discontinuum is the ideal arrangement of homogeneous individual particles connected by some kind of contact. This idealization of the natural substances may be attained as a spatial lattice or as a random distribution of particles. In the first case, the methods of the Crystallography are to be applied, regarding its high theoretical development (e.g. Klein and Hurlbut, 1996). In the second case, methods of the statistical mechanics may be applied (e.g. Aste 1996).

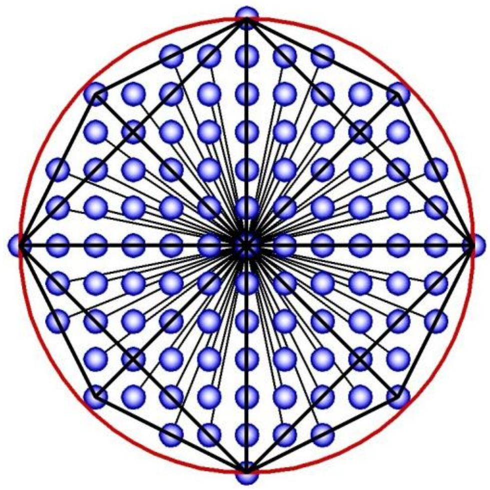

Figure 1. Plan and front views of the hexaoctahedron as an example of natural domain of influence, D, for a cubic or isometric spatial lattice.

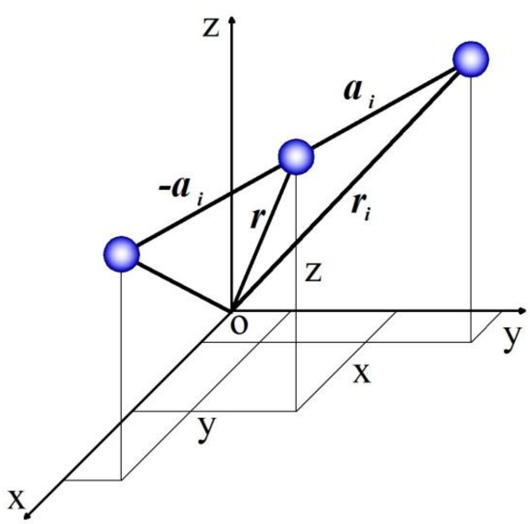

Figure 2. The homologous point **r** and the neighboring point $\mathbf{r_i}$ as the basis to calculate the mean value in a discontinuum.

A lattice is an infinite array of points in regularly and symmetrically way (Fig. 1). In this context, a point represents any thing, such as a particle or a void. In the whole, the lattice is homogeneous because any of its parts is neither worse nor better than all the other parts, so that its points are called homologous points. Likewise, the lattice is discrete and inherently anisotropic because it is made of individual points. As early as 1848, Bravais, showed that symmetry restricts the number of possible lattices to fourteen physically admissible arrays, grouped from the lower to the higher symmetry in triclinic, monoclinic, rhombic, tetragonal, hexagonal, and cubic or isometric.

To emphasise the discrete character of the lattice, it is customary to write the local or cell position vector of any homologous point, **a**, in terms of the three vectors, $\mathbf{b_1}$, $\mathbf{b_2}$, and $\mathbf{b_3}$, defined by the distance from one homologous point to the nearest neighbouring homologous points, and called the basic vectors. In Cartesian tensor notation: $\mathbf{a} = m_i\mathbf{b}_i$, where m_i are the integer Miller indices. However, for the discontinuum analysis, the cell position vector **a** shall be defined in terms of the Cartesian unit vectors as $\mathbf{a_i} = a_j\mathbf{i}_j$, being a_i proportional to m_j. In a general Cartesian coordinates system, the position vector of a point i, with respect to a known point, is written as $\mathbf{r_i} = \mathbf{r} + \mathbf{a_i}$, (Fig. 2).

2.2. The mean value principle

When the law that governs some phenomenon is unknown, or it is so complicated that is not suitable for practical application, the function or quantity f = f(**r**) may be assessed by applying the principle of the mean value, that states that the quantity or function f is equal to the average of all quantities or fuctions $f_i = f(\mathbf{r_i})$ given in a certain space, called the natural domain of influence of the homologous point **r**. Thus,

$$f(\vec{r}) = \bar{f}(\vec{r}_i) \quad [1]$$

The domain of influence can be defined as one of the Curie´s limit groups of symmetry, and is a polyhedron because of the discrete nature of the lattice (Fig. 1). The function $f(\mathbf{r_i}) = f(\mathbf{r}+\mathbf{a_i})$ can be expanded as a Taylor's series, in which odd terms become zero due to the symmetry of the lattice and terms of the higher order may be discarded. So that, the equation of the mean value (1) takes the form $(\upsilon_{jk}\partial^2_{jk})f = 0$; where υ_{jk} is the second-rank metric tensor of the polyhedron of influence D, and the expression in parenthesis, the generalized Laplacian operator. As long as this tensor is symmetric, there are three principal directions, upon which the coefficients υ_{jk}, for j≠k, disappear.

When the value of the function f(**r**) does not coincide with the mean value of f($\mathbf{r_i}$), there is a difference between them, known in statistics as the mean deviation, that is given by $\delta = (\upsilon_{jk}\partial^2_{jk})f$. But $\delta = \delta(\mathbf{r})$ may be considered itself as a function of state, and may be determined by means of the mean value as $(\upsilon_{jk}\partial^2_{jk})\delta = 0$. So that, an equation of higher approximation for the function f is established: $(\upsilon_{jk}\partial^2_{jk})^2 f = (\upsilon_{jk}\upsilon_{rs}\partial^4_{rsjk})f = 0$. By repeating this reasoning, the equation of the mean value of order n is achieved as

$$\left(\upsilon_{jk}\partial^2_{jk}\right)^n f = 0 \quad [2]$$

where the metric tensor of the polyhedron of influence, made of N points, is given by the expression:

$$\upsilon_{jk} = \frac{1}{2N} \sum_{i=1}^{N} a_{ji} a_{ki} \qquad [3]$$

Equation (2) describes the distribution of the quantity f in natural orderly matter and overcomes the undesirable dual approach to describe it (e.g. Sirotin and Shaskolskaya, 1982). Up to now, matter was considered inherently discontinuous in discussing its internal structure, but continuous when describing the physical laws that govern its behavior. In this path, it should be noted that the averaging of the first order in an isometric discontinuous substance becomes the Laplace´s equation: $\partial^2_{kk}f = 0$, that describes the behavior of an isotropic matter since it does not depend on the metric tensor of the lattice.

2.3. The biased mean value

As long as a discontinuous matter is made of discrete units, it is possible to obtain approximate but simpler solutions to a boundary value problem, considering a partial or biased domain of influence, B, consisting of a set of points chosen conveniently (Yanqui 2008). Statistically, B is a sample of the population D, and, because of this, the function of state can be only assessed by an estimator, which hereafter shall be denoted by g = g(**r**). Two features are important in applying the mean value principle to this estimator. One is the modification of the average metric tensor because there are a deficient number of points within the influence domain, B, and the other is the establishment of a rule to calculate g, by knowing that it does not fulfil the requirements of the principle of the mean value. Namely, if the average of the values of the estimator inside B, is bigger than g, there is an error *e* in the estimation of g by means of $\bar{g}$, at every point **r**. So that, the function g can be attained correcting the mean value $\bar{g}$. According to the kind of error, the principle of the mean value is stated as $g = \bar{g} - e$, for an error independent of g, or as $g = \bar{g} - e_s g$, for a systematic error. For the first case, $\tilde{\upsilon}_{jk}\partial^2_{jk}g = e$, and for the second case, $(\tilde{\upsilon}_{jk}\partial^2_{jk})g = e_s g$, or, in symbolic way: $(\tilde{\upsilon}_{jk}\partial^2_{jk} - e_s)g = 0$, where $\tilde{\upsilon}_{jk}$ is given by the equation (3), provided that the number of points involved are Ñ.

The arguments put forward for the function of state, f , with respect to the principle of the mean value of higher order, are completely valid for the function of estimation g. Applying n times this operation, the differential equation of the biased mean value of order n is written as

$$\left(\tilde{\upsilon}_{jk}\partial^2 jk - e_s\right)^n g = 0 \qquad [4]$$

For instance, if the biased domain of influence is made of the points in the OXY plane, the only non vanishing components of the metric tensor for an isotropic medium are $\tilde{\upsilon}_{11} = \tilde{\upsilon}_{22} = \tilde{\upsilon}$; then, the equation of the mean value for the estimator g attains the hyperbolic fashion: $(\Delta-\beta^2)^n g=0$, where Δ is the two-dimensional Laplacian operator, and $\beta^2 = e_s/\tilde{\upsilon}$, the factor of correction.

3 STRESSES IN ISOMETRIC DISCONTINUUM

For an isometric lattice, the mean value of the second order yields the biharmonic equation. Thus,

$$\left(\partial^2_{kk}\right)^2 f = 0 \qquad [5]$$

This equation has been used to solve the problem on the semi-infinite medium by Boussinesq (1885), as well as the problem on the layered system by Burmister (1943), both in a full rigorous fashion. But, even in the simplest case of a uniformly distributed loading over a circular area, the solution is too complicated. So that, for the sake of practical usefulness, a solution approximate but rigorous enough should be looked for. As one of the most elementary approximations, it is considered an influence domain, biased in the three Cartesian directions not only with respect to the gathering of the homologous points but also with respect to the Hooke´s law, that are written as

$$\sigma_x = K\varepsilon_x \;; \quad \sigma_y = K\varepsilon_y \;; \quad \sigma_z = K\varepsilon_z \qquad [6]$$

3.1 Biased displacements and stresses

The principle of the mean value and, even more, the biased dominium of influence are statistical tools that allow the deduction of several expressions for the same phenomenon. In this case, a stress function of the Kirchhoff´s kind, which coincides with the modified stress function of the Love´s kind, is chosen to be used, because it satisfies the differential equations of equilibrium, the compatibility conditions and the Hooke´s law, biased in the three Cartesian directions. Hence:

$$u = -\frac{\partial^3 g}{\partial x \partial z^2} \qquad [7]$$

$$v = -\frac{\partial^3 g}{\partial y \partial z^2} \qquad [8]$$

$$w = \frac{\partial}{\partial z}\left(\frac{\partial^2 g}{\partial z^2} + \Delta g\right) \qquad [9]$$

$$\sigma_x = -K\frac{\partial^4 g}{\partial x^2 \partial z^2} \qquad [10]$$

$$\sigma_y = -K\frac{\partial^4 g}{\partial y^2 \partial z^2} \qquad [11]$$

$$\sigma_z = -K\Delta^2 g = K\frac{\partial^2}{\partial z^2}\left(\frac{\partial^2 g}{\partial z^2} + 2\Delta g\right) \qquad [12]$$

$$\tau_{xy} = -K\frac{\partial^4 g}{\partial x \partial y \partial z^2} \qquad [13]$$

$$\tau_{xz} = K \frac{\partial^2}{\partial x \partial z} (\Delta g) \qquad [14]$$

$$\tau_{yz} = K \frac{\partial^2}{\partial y \partial z} (\Delta g) \qquad [15]$$

3.2 Method of separation of variables

If the stress function is given by the expression g = Zφ =Z(z)φ(x,y), the biharmonic equation may be simplified to the following much simpler differential equations:

$$Z^{IV} - 2\beta^2 Z^{II} + \beta^4 Z = 0 \qquad [16]$$

$$\Delta\varphi = -\beta^2\varphi \qquad [17]$$

$$\Delta^2\varphi = \beta^4\varphi \qquad [18]$$

which can be achieved also by separately application of the principle of the mean value to each variable Z(z) and φ(x,y). As mentioned before, these operations are possible only because of the discrete nature of the lattice. The first equation is accomplished by applying the mean value principle of the second order to a biased domain of influence made of a vertical row of particles. The other two equations are obtained by virtue of the mean value principle of the first and the second order, respectively, applied to a horizontal net of particles. In both cases, the error of estimation is considered systematic. Substitution of the separable variables into equations (7) to (15) yields the following expressions for the displacement vector and the stress tensor:

$$u = -Z'' \frac{\partial\varphi}{\partial x} \qquad [19]$$

$$v = -Z'' \frac{\partial\varphi}{\partial y} \qquad [20]$$

$$w = -\frac{1}{\beta^2}(Z''' - 2\beta^2 Z')\Delta\varphi = (Z''' - 2\beta^2 Z')\varphi \qquad [21]$$

$$\sigma_x = -KZ'' \frac{\partial^2\varphi}{\partial x^2} \qquad [22]$$

$$\sigma_y = -KZ'' \frac{\partial^2\varphi}{\partial y^2} \qquad [23]$$

$$\sigma_z = -KZ\Delta^2\varphi = K\beta^2 Z\Delta\varphi \qquad [24]$$

$$\tau_{xy} = -KZ'' \frac{\partial^2\varphi}{\partial x \partial y} \qquad [25]$$

$$\tau_{xz} = KZ' \frac{\partial}{\partial x}(\Delta\varphi) = -K\beta^2 Z' \frac{\partial\varphi}{\partial x} \qquad [26]$$

$$\tau_{yz} = KZ' \frac{\partial}{\partial y}(\Delta\varphi) = -K\beta^2 Z' \frac{\partial\varphi}{\partial y} \qquad [27]$$

4 THE MULTILAYER SYSTEM

The method of separation of variables is suitable to solve important geotechnical problems, such as the settlement of layered soils (e.g. Harr, 1966), pavement analysis (e.g. Huang, 1993), and so forth. For them, the dissipation of stresses and displacements caused by a distributed vertical load, applied on the surface of a system made of horizontal layers of different mechanical characteristics, is the main target. Downward local coordinates are used, with the origin at the central upper level of each layer, whose individual features are: a thickness H, a modulus of proportionality E, and a Poisson ratio v. Each layer is nominated by a number i, counted from the top to the bottom, being N the number of layers. All of them are finite except the bottom layer N that deepens infinitely (Fig. 3).

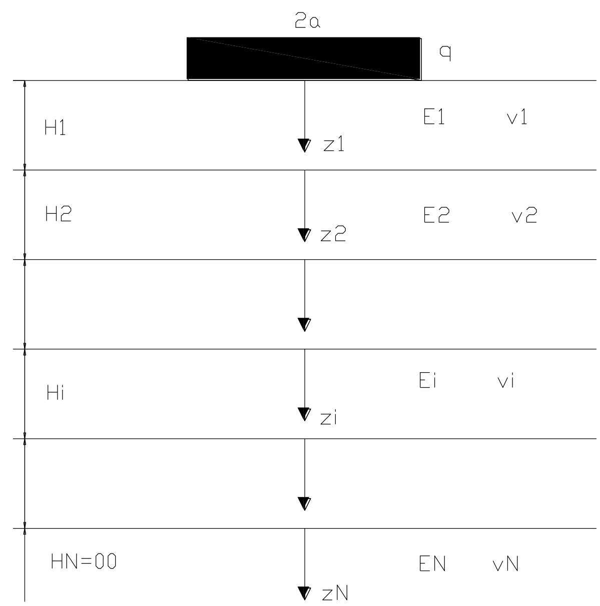

Figure 3. Definition of parameters in the multilayer system.

4.1. Solution for the vertical direction

The solution of the linear differential equation (16) involves hyperbolic sines and cosines. So that, a generalized solution is achieved by adopting the following nomenclature: $fh_m\beta z = \cosh\beta z$, if m is even, and $fh_m\beta z = \sinh\beta z$, if m is odd. Then, for the layer i, the derivative of the n-th order of the vertical function is given by

$$Z_i^{(n)} = \beta^{n-1}\{[\beta(A_i + C_i z_i) + nD_i] fh_{n-1}\beta z_i + [\beta(B_i + D_i z_i) + nC_i] fh_n \beta z_i\} \qquad [28]$$

If two successive layers are bonded, the condition of continuity requires that at the interface, the shear stress, vertical stress, radial displacement, and vertical displacement be equal. In a symbolic form:

$$(\tau_{rz})_{i+1} = (\tau_{rz})_i \quad [29]$$

$$(\sigma_z)_{i+1} = (\sigma_z)_i \quad [30]$$

$$(u)_{i+1} = (u)_i \quad [31]$$

$$(w)_{i+1} = (w)_i \quad [32]$$

These four conditions yield an explicit system of equations that relates the constants of integration of the successive layers i and i+1. They can be written as:

$$\begin{Bmatrix} A_{i+1} \\ B_{i+1} \\ C_{i+1} \\ D_{i+1} \end{Bmatrix} = \begin{bmatrix} t_{i11} & t_{i12} & t_{i13} & t_{i14} \\ t_{i21} & t_{i22} & t_{i23} & t_{i24} \\ t_{i31} & t_{i32} & t_{i33} & t_{i34} \\ t_{i41} & t_{i42} & t_{i43} & t_{i44} \end{bmatrix} \begin{Bmatrix} A_i \\ B_i \\ C_i \\ D_i \end{Bmatrix} \quad [33]$$

where the nucleus of this equation is the local transfer matrix i , in which the following notation is used, just for serendipity:

$$s_i = \sinh \beta H\,;\; c_i = \cosh \beta H\,;\; \lambda_i = \frac{K_i}{K_{i+1}} \quad [34]$$

So that, the terms of this matrix are written as:

$$t_{i11} = \tfrac{1}{2}(\lambda_i + 1)c_i \quad [35]$$

$$t_{i12} = \tfrac{1}{2}(\lambda_i + 1)s_i \quad [36]$$

$$t_{i13} = \tfrac{1}{2\beta}\left[(\lambda_i + 1)\beta H_i c_i + (\lambda_i - 1)s_i\right] \quad [37]$$

$$t_{i14} = \tfrac{1}{2\beta}\left[(\lambda_i + 1)\beta H_i s_i + (\lambda_i - 1)c_i\right] \quad [38]$$

$$t_{i21} = \lambda_i s_i \quad [39]$$

$$t_{i22} = \lambda_i c_i \quad [40]$$

$$t_{i23} = \lambda_i H_i s_i \quad [41]$$

$$t_{i24} = \lambda_i H_i c_i \quad [42]$$

$$t_{i31} = -t_{i42} = -\tfrac{1}{2}(\lambda_i - 1)\beta s_i \quad [43]$$

$$t_{i32} = -t_{i41} = -\tfrac{1}{2}(\lambda_i - 1)\beta c_i \quad [44]$$

$$t_{i33} = -\tfrac{1}{2}\left[(\lambda_i - 1)\beta H_i s_i - 2c_i\right] \quad [45]$$

$$t_{i34} = -\tfrac{1}{2}\left[(\lambda_i - 1)\beta H_i c_i - 2s_i\right] \quad [46]$$

$$t_{i43} = \tfrac{1}{2}\left[(\lambda_i - 1)\beta H_i c_i + (\lambda_i + 1)s_i\right] \quad [47]$$

$$t_{i44} = \tfrac{1}{2}\left[(\lambda_i - 1)\beta H_i s_i + (\lambda_i + 1)c_i\right] \quad [48]$$

Orderly and sequential application of equation (33) to all layers of the system leads to the equation that relates the constants of integration of the first and the last layer:

$$\begin{Bmatrix} A_N \\ B_N \\ C_N \\ D_N \end{Bmatrix} = [T] \begin{Bmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \end{Bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} & T_{14} \\ T_{21} & T_{22} & T_{23} & T_{24} \\ T_{31} & T_{32} & T_{33} & T_{34} \\ T_{41} & T_{42} & T_{43} & T_{44} \end{bmatrix} \begin{Bmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \end{Bmatrix} \quad [49]$$

where the global transfer matrix is attained as the systematic product of all local transfer matrices:

$$[T] = \prod_{i=N-1}^{1} \begin{bmatrix} t_{i11} & t_{i12} & t_{i13} & t_{i14} \\ t_{i21} & t_{i22} & t_{i23} & t_{i24} \\ t_{i31} & t_{i32} & t_{i33} & t_{i34} \\ t_{i41} & t_{i42} & t_{i43} & t_{i44} \end{bmatrix} \quad [50]$$

Now, the external boundary conditions can be established at the top and the bottom of the layer system. At the surface of layer 1, for $z_1=0$: $Z_1=1$ and $Z_1'=0$, due to the normalization of the vertical stress and the absence of the shear stress. These result in: $B_1=1$ and $D_1=-\beta A_1$. At layer N, for $z_N=\infty$: $Z_N=0$ and $Z_N'=0$ due to the full dissipation of all stresses. Then: $A_N+B_N=0$ and $C_N+D_N=0$. Substituting the values given by (49) into these equations yields all constants of integration for layer 1:

$$A_1 = \frac{b_2 b_3 - b_1 b_4}{b_4 b_5 - b_2 b_6} \quad [51]$$

$$B_1 = 1 \quad [52]$$

$$C_1 = \frac{b_1 b_6 - b_3 b_5}{b_4 b_5 - b_2 b_6} \quad [53]$$

$$D_1 = -\beta A_1 \quad [54]$$

where:

$$b_1 = T_{12} + T_{22} \quad [55]$$

$$b_3 = T_{32} + T_{42} \quad [56]$$

$$b_2 = T_{13} + T_{23} \quad [57]$$

$$b_4 = T_{33} + T_{43} \quad [58]$$

$$b_5 = T_{11} + T_{21} - \beta(T_{14} + T_{24}) \quad [59]$$

$$b_6 = T_{31} + T_{41} - \beta(T_{34} + T_{44}) \quad [60]$$

Once these values are known, the constants of integration of any layer j are obtained readily by means of the formula:

$$\begin{Bmatrix} A_j \\ B_j \\ C_j \\ D_j \end{Bmatrix} = \prod_{i=j-1}^{1} \begin{bmatrix} t_{i11} & t_{i12} & t_{i13} & t_{i14} \\ t_{i21} & t_{i22} & t_{i23} & t_{i24} \\ t_{i31} & t_{i32} & t_{i33} & t_{i34} \\ t_{i41} & t_{i42} & t_{i43} & t_{i44} \end{bmatrix} \begin{Bmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \end{Bmatrix} \quad [61]$$

4.2. Solution for the horizontal direction

Up to this point, the separable variable Z has been set down. Turning back to the boundary condition at the top of system, the vertical stress must be equal to the applied load: $\sigma_{z1} = -q(x,y)$, for $z_1 = 0$. Provided that it was assumed that $Z_1(0) = 1$, equation (24) yields the following differential equation for the other separable variable, $\varphi=\varphi(x,y)$:

$$\Delta\varphi = -\frac{q(x,y)}{\beta^2 K_1} \quad [62]$$

that has at least two important features. It simplifies equation (17) by changing the systematic error $\beta^2\varphi$ to an independent error $q/(\beta^2 K_1)$ in a proper way as long as q is small as compared with K_1. It describes the deflection of a plate strained by shear, and, therefore, the modulus *K* turns to be: $K=E/(1-v^2)$.

5 EQUIVALENT LOADED CIRCULAR AREA

For the beginning, an equivalent circular area for a loaded area of any convex shape is considered. Then, equation (62) can be written in an equivalent system of polar coordinates, and, therefore, integrated easily for a uniformly distributed load over a circular area of radius a_o:

$$\varphi = -\frac{qr^2}{4\beta_o^2 K_1} + c_1 \ln r + c_2 \quad [63]$$

where β_o is the equivalent coefficient of dissipation, and c_1 and c_2 , constants of integration to be determined by the boundary conditions in the horizontal direction. First, the finiteness of the vertical displacement at the center of the loaded area, requires that $c_1 = 0$. Second, the constant c_2 may be determined in such a way that equation (17) be satisfied by equation (63) at least at the origin of coordinates. Hence

$$\varphi = \frac{q}{\beta_o^4 K_1}\left(1 - \frac{1}{4}\beta_o^2 r^2\right) \quad [64]$$

Now, the non vanishing components of the displacement vector and the stress tensor for the layer i take the final form:

$$\sigma_{ri} = \sigma_{\theta i} = \frac{q}{2\beta_o^2}\left(\frac{K_i}{K_1}\right) Z_i'' \quad [65]$$

$$\sigma_{zi} = -q\frac{K_i}{K_1} Z_i \quad [66]$$

$$\tau_{rzi} = \frac{qr}{2}\left(\frac{\beta}{\beta_o}\right)^2 \left(\frac{K_i}{K_1}\right) Z' \quad [67]$$

$$u_i = \frac{qr}{2\beta_o^2 K_1} Z_i'' \quad [68]$$

$$w_i = \frac{q}{\beta_o^4 K_1}\left(1 - \frac{1}{4}\beta_o^2 r^2\right)\left(Z_i''' - 2\beta^2 Z_i'\right) \quad [69]$$

6 LOADED CIRCULAR AREA

Obviously, for a loaded circular area: β=βo, and equations (65) to (69) get a simpler form. This is the most known solution for a multilayer system.

6.1. Solution for a semi-infinite medium

For the one layer system, N = 1 and i = 1. Therefore, the stresses and displacements under the loaded circle are given by the expressions:

$$\sigma_r = \sigma_\theta = -\frac{q}{2}(1 - \beta z)e^{-\beta z} \quad [70]$$

$$\sigma_z = -q(1 + \beta z)e^{-\beta z} \quad [71]$$

$$\tau_{rz} = -\frac{qrz}{2}\beta^2 e^{-\beta z} \quad [72]$$

$$u = -\frac{qr}{2K}(1 - \beta z)e^{-\beta z} \quad [73]$$

$$w = \frac{q}{\beta K}\left(1 - \frac{\beta^2 r^2}{4}\right)(2 + \beta z)e^{-\beta z} \quad [74]$$

which are to be compared with the results of the Boussinesq´s theory. The coefficient of stress dissipation β may be assessed at once by equating the maximum settlement obtained from equation (74) for z=0 and r=0, to the well known formula: $w_0 = 2aq(1-v^2)/E$. Since β must be independent of any feature of the layer, two relationships show up:

$$\beta = \frac{1}{a}; \qquad K = \frac{E}{1-\nu^2} \qquad [75]$$

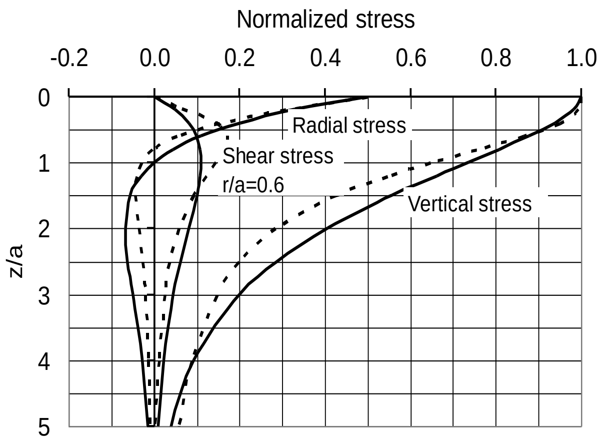

Figure 3. Stresses in a semi-infinte medium. Continuous lines stand for equations (34) and (35), and dots lines, for stresses given by the Boussinesq´s theory.

For a multilayer system, the first of them remains constant, and the second changes with the layer. In figure 3, it is shown the good correspondence of the stresses given by equations (70), (71) and (72) and those given the Boussinesq´s rigorous theory, revealing a maximum difference for the vertical stress of $0.134\sigma_z/q$ at a depth of 1.5z/a.

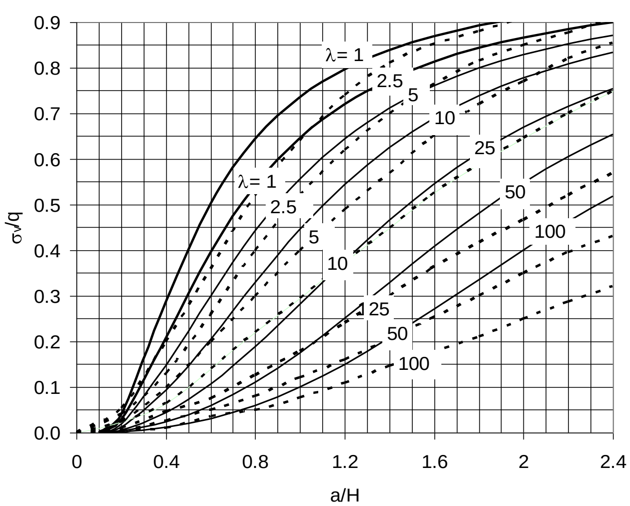

Figure 4. Vertical normal stress at the interface in a two-layer system. The continuous line stand for equation (42) and the dots line for the results from the Burmister´s theory.

6.2. Solution for a two-layer system

For a layer underlain by a flexible base: N=2 and stresses and displacements for i=2 are the same as those given by equations (70) to (74), provided that q is substituted by:

$$\sigma_{v2} = -q[(\ s_1 - \beta H c_1)A_1 + s_1 H C_1 + c_1] \qquad [76]$$

where H is the thickness of the upper layer, and A_1 and C_1 are given by equations (51) and (53). In figure 4, it can be seen that the vertical stress at the interface given by equation (76) are a little bigger than that found by Burmister. This happens because the horizontal distribution of the vertical stress has a shape of a step function, for the biased discontinuum mechanics, and a bell shape, for the theory of elasticity. However, it is interesting to notice that for both smaller and bigger values of the ratio a/H, this difference reverts.

6.3. Solution for a three-layer system

Even though it is possible to write explicit expressions for a three-layer system by using equations (65) to (69), it is better to asses them by a computer program. In table 1, some values of the vertical stress at the interfaces are shown.

Table 1. Comparison between equation (66) and Burmister´s solution (B),(Huang, 1993), for K_2/K_3=2.

H_1/H_2	a/H_2	K_1/K_2	σ_{v2}/q (B)	σ_{v2}/q (66)	σ_{v3}/q (B)	σ_{v3}/q (66)
0.25	0.1	2	0.1552	0.2292	0.0071	0.0003
0.25	0.4	2	0.7794	0.8226	0.1031	0.1207
0.25	1.6	2	0.9816	0.9760	0.6675	0.7254
1.00	0.1	2	0.0108	0.0004	0.0024	0.0001
1.00	0.4	2	0.1466	0.2261	0.0372	0.0225
1.00	1.6	2	0.7103	0.7843	0.3869	0.5057
0.25	0.1	20	0.0438	0.0536	0.0053	0.0001
0.25	0.4	20	0.3788	0.5794	0.0793	0.0687
0.25	1.6	20	0.9874	0.9423	0.6167	0.6152
1.00	0.1	20	0.0026	0.0000	0.0010	0.0000
1.00	0.4	20	0.0381	0.0527	0.0156	0.0044
1.00	1.6	20	0.3157	0.5074	0.2010	0.2991

7 LOADED AREA OF ARBITRARY SHAPE

For an area other than the circular one, the coefficient of dissipation of the stress, β, is to be found by comparison with some known solution, which, as a matter of fact, is very complicated or does not exist. Therefore, it is necessary to find an approximate way to calculate it, to be consistent with the current simplified theory. To accomplish it, the variable Z must be eliminated from equations (66) and (67), for r = a, to set down the equation:

$$\frac{d\sigma_z}{dz} + \frac{2}{a}\tau_{rza} = 0 \qquad [77]$$

which stands for the equilibrium in the vertical direction of a cylindrical element of radius a and thickness dz located at a depth z. In the same way,

$$\frac{d\sigma_z}{dz} + \frac{P}{S}\tau_{rze} = 0 \qquad [78]$$

is the equilibrium equation for a cylindrical element of arbitrary area S, perimeter P, and thickness dz. Writing σ_z in terms of the displacement w, and substituting equation (67), for $r=1/\beta_o = (S/\pi)^{1/2}$, and equation (74), for r=0, into (76), the following equation is met:

$$\beta = \frac{\pi^{1/4} P^{1/2}}{2^{1/2} S^{3/4}} \qquad [79]$$

For instance, with respect to the equation (69), the shape factor of the maximum vertical displacement in a rectangular area of sides 2a and 2b, may be written as:

$$I_S = \frac{2}{\pi^{1/4}} \frac{m^{3/4}}{(1+m)^{1/2}} \qquad [80]$$

where m=b/a. Figure 4 shows the excellent agreement with the shape factor of the Boussinesq´s theory.

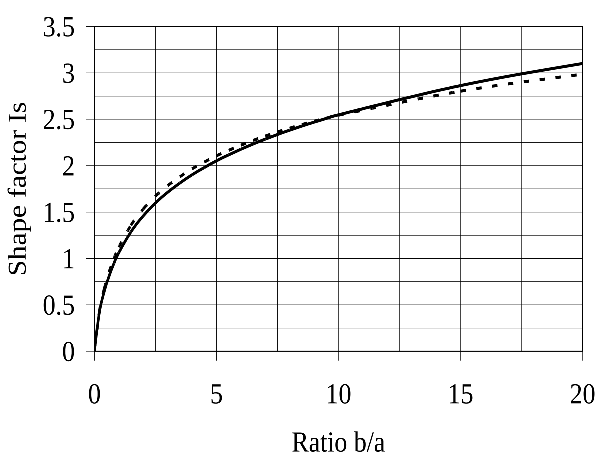

Figure 5. Shape factor for a loaded rectangular area. The continuous line stand for equation (79) and the dots line for the results from the theory of elasticity.

CONCLUSIONS

Deterministic Discontinuum Mechanics is based upon the structure of the matter and the principle of the mean value. The first is better described by the Bravais´s spatial lattices. The second is a general, simple and direct approach to describe any physical phenomenon. In particular, the principle of the mean value serves to solve the problem of the stress transmission in a layered soil, by averaging the Kirchhoff´s stress function in a biased domain of influence. As a consequence of this operation, the components of the displacement vector and stress tensor are expressed as functions of the elements of the transfer matrices of the fourth order. The separable variable in the horizontal direction reduces finally to a second degree polynomial. An approximate solution for a uniform load applied over an area of any shape is obtained by the appropriate definition of the parameters of equivalence to the circular loaded area. The solutions so found fit well with those of the theory of elasticity, for isotropic materials.

REFERENCES

Aste, T., 1996. Circle, sphere, and drop packings. *Physical Review* B, Vol. 53, No. 3: 2571-2579.

Aste, T., T. Di Matteo and E. Galleani d´Agliano, 2002. Stress transmission in granular matter. *Journal Phys. Condensed Matter*, Vol. 14: 2391-2402.

Bouchaud, J.P., 2002. Granular media: some ideas from statistical physics*. Journal of Condensed Matter*. Vol. 2/0211196.

Boussinesq, J. 1985. *Application des Potentials a L´Etude de L´Equilibre et du Mouvementdes Solides Elastiques*. Gauthier-Villars, Paris.

Burmister, D.M., 1943. The theory of stresses and displacements in layered systems and applications to the design of airport runways. *Proc. Highway Research Board*. Vol. 23.

Fung, Y.C., 1965. *Foundations of Solid Mechanics*. Prentice-Hall. New Jersey.

Harr, M.E., 1966. *Foundations of Theoretical Soil Mechanics*. McGraw Hill. New York.

Huang, Y.H. 1993 *Pavement Analysis and Design.* Prentice Hall. New Jersey.

Klein, C. and Hurlbut, JR. C.S. 1996. Manual of Minerology. John Wiley & Sons. New York.

Sirotin, Y.I. and Shaskolskaya, M.P. 1982 *Fundamentals of Crystal Physics*. MIR Publishers. Moscow.

Trollope, D.H., 1956. *The stability of wedges of granular materials*. PhD. thesis. University of Melbourne.

Yanqui, C. 1980. Clastic stresses in infinite slopes. *Proc. 3rd Peruvian Conference on Civil Engineering*: 234-248. Cusco. Peru.

Yanqui, C. 1982. *Statics of gravitating discontinua*. MSc. thesis, The University of South Carolina. USA.

Yanqui, C., 1995. Stresses induced by footings in discontinuous media. *10th Panamerican Conf. on Soil Mechanics and Foundations*. Vol. 3: 1341-1353. Guadalajara, Mexico.

Yanqui, C. 2008. Simplification of the Burmister´s problem by means of skew functions of discontinuum mechanics. *12th Int. Conf. on Computer Methods and Advances in Geomechanics*. 546-554. Goa, India.