# The theory of granular packings as a chapter of the soil mechanics subject

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ABSTRACT
Soils are discontinuous substances made of individual solid particles and voids, defined by their contacts. Therefore, the natural description of soils is better accomplished by using granular packings. In this context, to achieve the maximum mathematical simplicity, a grain is represented by a sphere whose diameter is equal to the centroidal distance between two grains in contact, and the fundamental element, by the parallelepiped drawn by the centroides of eight neighbouring spheres, which must be in statically equilibrium, whose solid volume is obviously equal to the volume of the effective grain, and whose geometric parameters define the structure of the assembly, as well as the total volume. For the simplest granular packing, the Wadell´s shape coefficient and the Hazen´s uniformity coefficient are redefined. Under these considerations, it is settled down the fundamental equation that relates any global parameter, such as the void ratio or the volume ratio, with the structural parameters of the element. This connection is used to explain and calculate the physical and mechanical properties of soils; for instance, the relationship between the densest state and the loosest state, the relationship between the angle of internal friction and the coefficient of lateral stress "at rest", the relationship between that angle and the porosity, and the value of the Casagrande´s critical void ratio, among others. All relationships so obtained fit very well with the experimental data reported by worldwide authors.

RESUMEN
Los suelos son sustancias discontinuas conformadas por partículas sólidas y poros, definidos por sus contactos. En consecuencia, la descripción natural de los suelos se consigue de una mejor manera usando los ensambles granulares. En este contexto, para conseguir la máxima simplificación matemática, un grano se representa por una esfera cuyo diámetro is igual a la distancia centroidal entre dos granos en contacto y el elemento fundamental, por el paralelelpípedo dibujado por los centroides de ocho esferas vecinas, las cuales deben estar en equlibrio estático, cuyo volumen es obviamente igual al volumen del grano efectivo, y cuyos parámetros geométricos definen la estructura del ensamblaje, así como el volumen total. Para una ensamble granular simple, se redefinen el coeficiente de forma de Wadell y el coeficiente de uniformidad de Hazen. Bajo estas consideraciones, se establece la ecuación fundamental que relaciona cualquier parámetro global, como el índice de poros o el indice volumétrico con los parámetros estructurales del elemento. Esta relación se usa para explicar y calcular las propiedades físicas y mecánicas del suelo; por ejemplo, la realación entre los estados más denso y más suelto, la relación entre el ángulo de rozamiento interno y el coeficiente de esfuerzo lateral « en reposo », la relación entre aquel ángulo y la porosidad, y el valor del índice de poros crítico de Casagrande, entre otras. Todas estas relaciones así obtenidas se ajustan bien a los datos experimentales reportados por autores de todo el mundo.

## 1 INTRODUCTION

”Soil is inherently a particulate system. Indeed, the science that treats the stress-strain behavior of soil may well be thought of as particulate mechanics” (Lambe and Whitman, 1969). “The way out of the difficulty lies in dropping the old fundamental principles and starting again from the elementary fact that the sand consists of individual grains.” (Terzaghi, 1920). Statements of this kind have been made several times by prominent authors. Therefore, it is compulsory to introduce some granular model in the Soil Mechanics to grasp its particulate nature. Being a branch of the physical science, this model must exhibit three merits: comprehensiveness, predictive power and simplicity (e.g. Brancazio, 1975). Within this frame of reference, a new chapter of the soil mechanics subject is proposed to rationally explain the changes of porosity, the extreme states of compactness, the transmission of simple stresses, the shear strength, and the critical state of granular soils. But the model to be outlined in this report is applicable to fine soils as well, to explain quantitatively the flocculation of silty grains, the polarization of clay sheets, the effect of the adsorbed double layer, the nature of Atterberg limits, among other topics that shall not be treated here.

## 2 THEORY OF GRANULAR PACKINGS

Soil is a discontinuous substance made of an assemblage of grains and pores. Grains are solid bodies arbitrary in size, shape, orientation and surface texture. Pores are the space where there are no grains and may contain air and water. Grains are interconnected through almost punctual contacts, forming a highly complex and generally random system, referred to as soil structure. From the practical point of view, two features are most important in the description of the soil: the shape and the gradation of the grains.

A granular packing is an orderly regular array of spheres of the same size and smooth surface texture. Lattice is the arrangement of the centers of the spheres, called homologous points, and obeys the laws of symmetry of crystals, for which, Bravais demonstrated, as early as 1848, that can only have fourteen kinds physically acceptable. The oblique parallelepiped, constituted by eight homologous points, neighbors with each other, pertaining to the lattice is called the unit cell (Klein and Hurlbut, 1996). In this context, a fundamental assumption is stated: a soil can be modelled as an ideal granular packing made of spheres representatives of all real grains. This transformation can be achieved through a proper definition of their physical characteristics, called textural parameters.

2.1 Grain equivalent diameter

The equivalent diameter comes from the most basic transformation of a grain into a sphere, and has been proposed by many authors, mention apart that constitutes the fundamental working hypothesis in assessing the size of the fine soils, for example, when using the hydrometer. The equivalent diameter, D, is the diameter of the sphere of equal volume as the grain, $V_s$, (Fig. 1), thus

$$D = \sqrt[3]{\frac{6V_s}{\pi}} \qquad [1]$$

The equivalent diameter of coarse sand and gravel can be determined using the pycnometer method, whereby the volume of grain is equal to the volume of water displaced. The equivalent diameter of fine soil, obtained by the adsorption test, is defined as the diameter of the sphere of equal specific surface area, $S_s$. and equal weight, $\gamma_s$, to the grain

$$D = \frac{6}{S_s \gamma_s} \qquad [2]$$

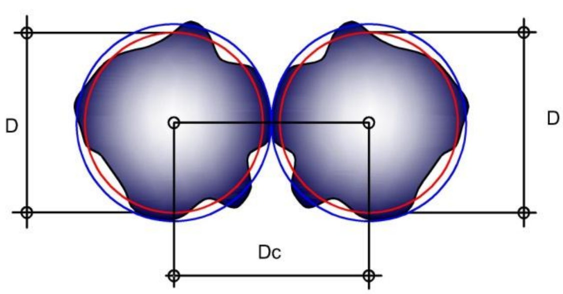


Figure 1. Equivalent diameter and diameter of contact for grains.

2.2. Contact diameter

The measure of the separation between the centroids of two grains is called contact diameter, $D_c$, (Fig.1) and can be determined for gravels by a simple test. For better statistical accuracy, several coarse grains are placed in a channel of semicircular section and the distance between the first and last grain is measured. This value is divided by the number of grains involved to get the contact diameter. Obviously, the grains should be of the same size and, at least in theory, the contact diameter is always greater than the equivalent diameter.

2.3. Coefficient of particle shape

Originally, Wadell defined the shape factor or sphericity of a grain as the ratio of grain surface area and surface of the equivalent sphere. But, due to practical difficulties, Wadell himself later amended this definition to the relationship between the volume of the grain and volume of the circumscribed sphere. In this theory, this coefficient is defined as the ratio of the volume of the grain and the volume of the sphere of contact

$$\xi_a = \frac{1}{\chi_a} = \frac{V_s}{V_c} = \left(\frac{D}{D_c}\right)^3 \qquad [3]$$

2.4. Generalized uniformity coefficient

The gradation of the soil is the second feature to be modified. It is said that a soil is uniform when consists of grains of equal size. To take into account the fact that the vast majority of soils are, on the contrary, graduated, Hazen defined the uniformity coefficient as the relationship between the diameter $D_{60}$ and the diameter $D_{10}$. The first represents approximately the average diameter of the soil and the second, the effective diameter. Therefore, the generalization of this factor leads to the following expression:

$$\chi_u = \left(\frac{D}{D_{ef}}\right)^3 \qquad [4]$$

where D is the structural diameter and $D_{ef}$, the representative integral diameter of the soil.

2.5. Textural grain coefficient

The two texture coefficients have the same effect on the packing: the content of pores increases with the angularity and uniformity of grains and decreases with the roundness and gradation of sizes. Since in general, it is not possible to discriminate the participation of each of them, is more practical to use a single textural grain coefficient:

$$\chi_g = \chi_u \chi_a \qquad [5]$$

2.6. Unit cell volume

The main merit of the unit cell is the universality of its volume, because, as it is well known, the volume of a parallelepiped is found by multiplying the area of the base by the height. This means that, whatever be the kind of lattice, the volume is met by knowing the dimensions and directions of the edges of the parallelepiped formed by eight neighboring homologous points. In a granular packing, the contact diameter is a constant quantity that depends on the textural characteristics of soil. The angle of the basal parallelogram α, and the angle that the generatrix makes with the vertical line, not only define the lattice structure, but allow the classification of granular packings. For example, considering the ease of representation, they are classified as two and three dimensional, and if it is considered the nature of the directrix, in prismatic and pyramidal.

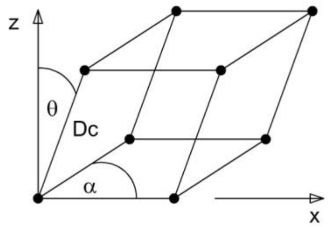


Figure 2. The lattice unit cell, and its geometrical elements.

2.6.1. Two-dimensional packings

Although in essence they are three dimensional, these packings can be represented in two dimensions due to its symmetry with respect to one of the Cartesian vertical planes. This means that the horizontal face and the oblique face are squares of side equal to the diameter of contact, while the vertical face is a parallelogram defined by the angle of the generatrix. In the nomenclature of Bravais, this lattice is called monoclinic, and geometrically corresponds to a parallelogram, which can be classified according to the location of the diameters of contact as equilateral and isosceles.

2.6.1.1 Equilateral parallelogram

In this case, all sides of the parallelogram are diameters of contact, so the volume is expressed in terms of the angle that the oblique side makes with the vertical, β. Then: $V=D_c^3\cos\beta$.

2.6.1.2. Isosceles parallelogram

In this type of packing, the horizontal sides of the parallelogram are not diameters of contact, owing to which a grain from the upper base must rest on the two grains of the lower base. So that, the oblique side is equal to the minor diagonal of the parallelogram and the angle with respect to the vertical is denoted by θ. Therefore, $V=D_c^3\sin 2\theta$.

2.6.2. Three-dimensional packings

In the most general configuration, three-dimensional packings require for its description of at least two angles. In the nomenclature of Bravais, this lattice is called triclinic and geometrically corresponds to a parallelepiped, which, like two-dimensional packings, can be classified as equilateral and tetrahedral.

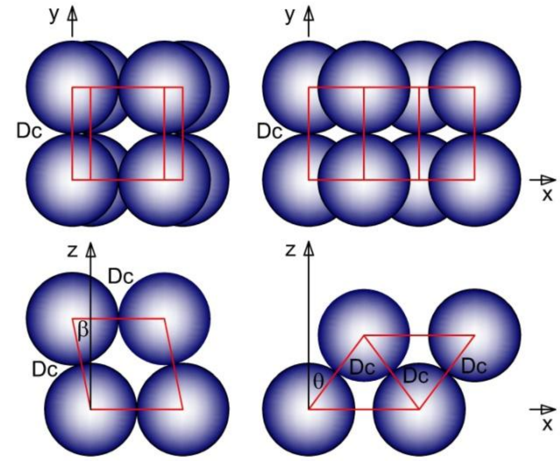


Figure 3. Plan and elevation view of the two-dimensional packing. a) Equilateral parallelogram in the OXZ plane and squares in the OXY plane. b) Isosceles parallelogram in the OXZ plane.

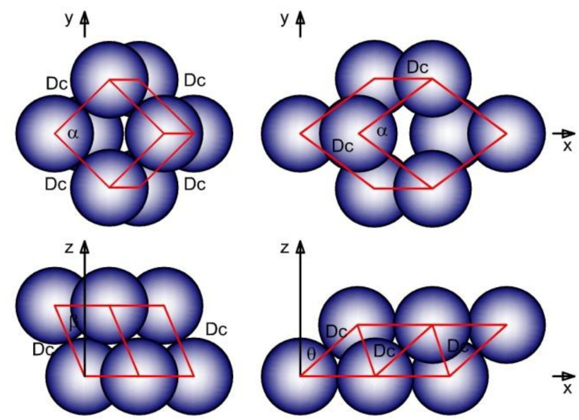


Figure 4. Plan and elevation view of the three-dimensional packing. a) Equilateral parallelepiped, b) Isosceles tetrahedral parallelepiped.

2.6.2.1 Equilateral parallelepiped

In this packing, all edges of the parallelepiped are diameters of contact, and each grain automatically satisfies the condition of static equilibrium. In this case, the angle of the generatrix with the vertical is denoted by β and the total volume is given by the expression: $V=D_c^3\sin\alpha\cos\beta$. According to the angle α, the base of this parallelepiped varies from a hexagonal rhomb to a square.

2.6.2.2 Tetrahedral parallelepiped

The directrix of this packing does not consist of contact diameters, and thus, every grain of the top layer is not in equilibrium, except if is supported by two grains of the layer below. When this occurs, the diagonals of the lateral faces of the parallelepiped are equal to the oblique edge and the parallelepiped is symmetric with respect to the plane containing the angle θ of the generatrix with the vertical. Then: $V=D_c^3\sin\alpha(1+\cos\alpha)\sin\theta\sin 2\theta$. According to the angle α, the base of this isosceles tetrahedron varies from an equilateral triangle to a square. In the last case, the tetrahedron transforms itself to an octahedron.

2.7 Solid phase volume in the unit cell

The second merit of the unit cell is the constancy of the volume of solids. It is also a known fact of elementary geometry that the sum of the eight octants formed by three oblique planes is equal to the total solid space, regardless of the angles α and θ. This principle also applies to a sphere and, even more, for any solid body. Indeed, the sum of the eight spherical trihedrons, defined by the faces of the parallelepiped is equal to the integral effective volume of the grain:

$$V_s = \frac{\pi}{6} D_{ef}^{\ 3} \qquad [6]$$

## 3 PHASE RELATIONSHIPS

Once the total volume and solid volume are known, the amount of pores that contain the packing may be determined. Historically, different definitions have been proposed, according to the need of the subject, for example, the porosity, which relates the pore volume to the total volume, $n = V_v/V$, the void ratio, e, which relates the pore volume with the volume of grains, $e = V_v/V_s$, or, more recently, the volumetric ratio, $\upsilon$, which relates the total volume to the volume of grains $\upsilon = V/Vs$, which leads to a greater compactness of the formulas (Wood, 1990). As they are all different expressions of the same thing, these parameters are related to each other:

$$\upsilon = e + 1 = \frac{1}{1-n} \qquad [7]$$

Hence, the volumetric ratio assumes a definite form for each type of packing:

Equilateral parallelogram: $$\upsilon = \frac{6}{\pi}\chi_g \cos\beta \qquad [8]$$

Isosceles parallelogram: $$\upsilon = \frac{6}{\pi}\chi_g \sin 2\theta \qquad [9]$$

Equilateral parallelepiped: $$\upsilon = \frac{6}{\pi}\chi_g \sin\alpha\cos\beta \qquad [10]$$

Tetrahedral parallelepiped: $$\upsilon = \frac{6}{\pi}\chi_g \sin\alpha(1+\cos\alpha)\sin\theta\sin 2\theta \qquad [11]$$

A quick inspection of these equations leads to the conclusion that, due to the nature of trigonometric functions, all of them accept two values. For example, the first equation is valid for β and -β, the second for θ and 90-θ and so on. Likewise, the equivalence of the two-dimensional packings themselves points out the following relationships:

$$\theta = 45^\circ - \beta/2 \quad \text{or} \quad \theta = 45^\circ + \beta/2 \qquad [12]$$

These facts illustrate one of the intrinsic properties of granular packings: their duality or, in a more general sense, their multiplicity.

3.1 Extreme states

The most important conclusion of the analysis developed up to this point is that the porosity of the granular packings changes according to the value of θ or α and β. The second conclusion concerns the restrictions imposed by the contact between grains, by which the porosity is a bounded quantity, called minimum porosity, $n_m$, and related to the densest state of the packing. In the two kinds of three-dimensional packing, the angle α can only take values between 60º and 90º. Furthermore, in the pyramidal packing, these extreme values are related to two axisymmetric lattices: the tetrahedral, for α = 60 °, and the octahedral for α = 90 °. Likewise, the angles β and θ, called structural angles, are bounded. In summary, the following ranges of validity are recognized: a) for the equilateral parallelogram: -30º ≤ β ≤ 30º; b) for the parallelogram isosceles: 30º ≤ θ ≤ 60º; c) for the equilateral parallelepiped: $-1/2\sec\rho \le \sin\beta \le 1/2\sec\rho$; $0 \le \rho \le \alpha/2$; 60º≤α≤90º; and d) for a tetrahedral parallelepiped: $\theta \ge \arcsin[1/2\sec(\alpha/2)]$, 60º≤α≤90º.

**Table 1.** Features of granular packings.

| Packing | Tetrahedral | Octahedral |
|---|---|---|
| Loosest State | $\cos\theta = \frac{1}{\sqrt{3}}$ | $\cos\theta = \frac{1}{\sqrt{3}}$ |
| Densest State | $\sin\theta = \frac{1}{\sqrt{3}}$ | $\sin\theta = \frac{1}{\sqrt{2}}$ |
| Minimum porosity | $n_m = 1 - \frac{\pi\sqrt{2}}{6\chi_g}$ | $n_m = 1 - \frac{\pi\sqrt{2}}{6\chi_g}$ |
| Maximum porosity | $n_M = 1 - \frac{\pi}{6\chi_g}$ | $n_M = 1 - \frac{\pi\sqrt{3}}{8\chi_g}$ |
| Extreme porosities Relationship | $n_m = 1.41 n_M - 0.4144$ | $n_m = 1.089 n_M - 0.087$ |

The third conclusion is related to the fact that each equation of the porosity accepts a mathematical root, which are related to the loosest state of the soil, and, therefore, to the maximum porosity, $n_M$, which separates the acute configuration from the obtuse configuration, and, therefore, is unique for each packing. The respective derivation yields the following results: β = 0, for the equilateral parallelepiped; θ = 45°, for the parallelogram isosceles and θ = arccos(1/√3) ≈ 54.74 ° for the tetrahedral parallelepiped. The substitution of these values and the corresponding extreme values for the densest state in equations (8) and (11) allows to establishing a relationship between the maximum porosity and the minimum porosity for each type of packing. In table 1, all values for the limit packings of the tetrahedral family are shown. Just for the sake of identification, they are named tetrahedral and octahedral

## 4. VALIDATION OF THE GRANULAR MODEL

The merit of the extreme states lies in the fact that they can be found experimentally by very simple testing. The maximum porosity is determined by the uniform pouring of the grains into a calibrated mold, and the minimum porosity, by the strongly penetration of a thin rod into the soil in the mold, by compacting it with a vertical hammer, by shaking it onto a vibrating table, or combining some of these procedures. In Figure 5, it is shown the experimental values for granular soils reported by various authors (Selig and Ladd, 1973), which can be compared with the theoretical values given by the extreme conditions for the tetrahedral and octahedral packings. It is worth to observe that the experimental data are in the domain bounded by the lines for the two axisymmetric pyramidal packings. The second conclusion drawn from this diagram is that the extreme porosities of a real granular substance are a measure not only of the pore volume but also of the grains shape and gradation: $\chi_g$. To the same conclusion arrived Talbot and Richart, as early as 1923, based on numerous experimental tests to obtain the densest state of coarse aggregates.

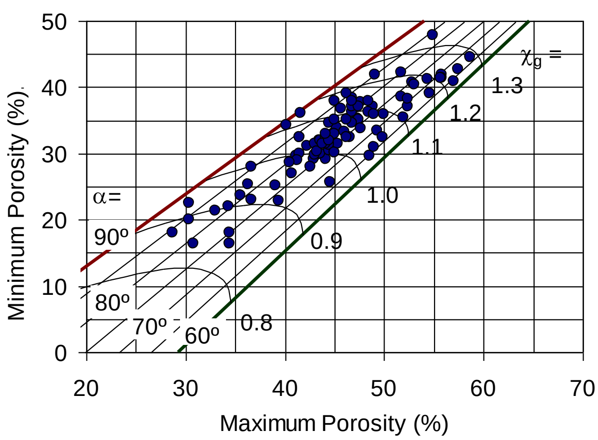


Figure 5. Maximum porosity versus minimum porosity diagram. The experimental data were reported by several authors. The lower straight line stands for the tetrahedral packing, and the higher, for the octahedral one.

## 5. MECHANICS OF SIMPLE GRANULAR PACKINGS

Unlike continuous media, the granular packings are prone to exhibit multiple mechanisms of transmission of the stress, depending on the nature of these and the contact points between grains. In the last two decades, random functions of the quantum mechanics have been used with some success to solve some particular problems (eg, Aste et al., 2002). Solutions so found are complicated and inaccessible from the point of view of engineering practice. However, the application of the principle of the mean value allows the deterministic and simple calculation of stress distribution under a general solicitation and boundary conditions (Yanqui, 1995). But the analysis of stress in a specimen subject to a uniform solicitation becomes an extremely simple task, if one accepts the principle of centroidal reactions (Trollope, 1956; Yanqui, 1980), whereby the contact lines of the unit cell coincide with the directions of contact forces. Some authors (e.g. Ostojic, 2006) have called this network of centroidal reactions a force network ensemble, which, however, does not necessarily coincide with the unit cell.

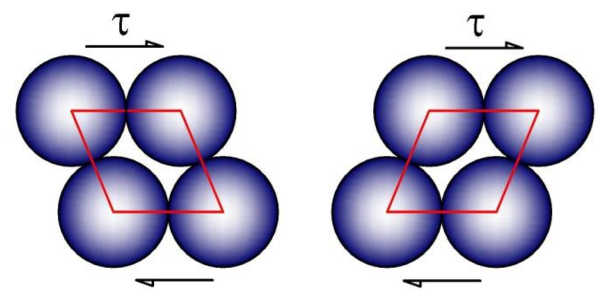


Figure 6. Mechanics of the shear stress. a) Dilatant element, b) Contractive element.

### 5.1. Simple shear

The prismatic packing is the best model to describe a simple shear test. In this case, all the edges of the unit cell are diameters of contact and, therefore, the forces ensemble coincides with the granular lattice (Fig. 6). The resultant of the vertical normal force N and the horizontal force T that supports a grain should be fully transmitted to the corresponding point of the lower layer, as long as in an element that works exclusively by shear the normal component in the horizontal edge is zero. Therefore, if the resultant coincides with the generatrix of the parallelepiped, this is a shear element that responds by diagonal compression. Another important aspect is the deformation of the granular packing. Being relatively rigid the spheres of contact, the displacement caused by the horizontal shear has a horizontal and a vertical component. The latter is related to so-called "dilatancy" of granular soils, analyzed first by Reynolds in 1885. But, because of the dual nature of the packing, also the opposite phenomenon may occur, which will be called "contractancy." In conclusion, considering the vertical axis OZ directed downward, the packing is dilatant when the sign of β is positive and the configuration is acute, while the packing is contractive when the sign of β is negative and the configuration is obtuse.

### 5.2 Two-dimensional compression

The two-dimensional simple compression test is described by a rhombic ensemble of forces by consideration of the horizontal symmetry (Fig. 7). In this case, the granular packing does not match the ensemble of forces, but the assemblage angle θ is the same for both, as well as the porosity. The principle of the centroidal reactions provides that a vertical force P applied to a grain of the upper layer is divided into two contact forces, F, symmetrical and oblique, whose magnitude is given by the expression: $F = P / (2\cos\theta)$. Therefore, the equilibrium in the horizontal direction requires a horizontal force of magnitude $Q = P\tan\theta$, or, in terms of stresses:

$$\sigma_{3i} = \sigma_1 \tan^2 \theta \qquad [13]$$

where $\sigma_1$ is the vertical stress and $\sigma_{3i}$, the internal horizontal stress, which must be compensated by applying a confining lateral stress of the same magnitude at least. This stress makes the difference with the continuum, which obeys the Cauchy´s principle, for which the stress is transmitted as if the substance were composed by wires parallel to the axis and independent of each other. In a granular medium, this mode of transmission of the stress is simply impossible because the pores of the packing generate oblique contact forces. Also, due to the dual nature of granular assemblies, there are two complementary values of the angle θ for the same void ratio. For θ less than 45 °, the contact lines coincide with the force ensemble. For θ greater than 45 º, the active state is possible only if the contact forces are not centroidal. Cinematically, in the first case, the element is dilatant and, in the second, contractive.

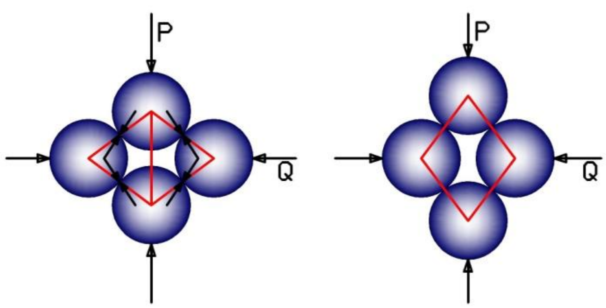


Figure 7. Mechanics of the two-dimensional confined compression. a) Contractive force ensemble, b) Dilatant force ensemble.

5.3 Triaxial compression

Regarding the axisymmetric character of this test, two force ensembles are possible: the rhombohedral and the octahedral, both equilaterals. In this case, the force ensembles do not coincide with the unit cell, except the angle, θ and the porosity. In this ensemble, the axial force P, acting at a grain, is decomposed into N contact forces, $F = P / (N\cos\theta)$, where N = 3 if it is a rhombohedron, and N = 4, if it is an octahedron (Fig. 8) In the first case, the problem is isostatic and in the second, hyperstatic. The transformation of this contact force to an average stress allows the calculation of the internal lateral stress:

$$\sigma_{3i} = \sigma_1 \frac{\tan^2 \theta}{2} \qquad [14]$$

that is balanced by the lateral pressure applied to the specimen. Although the analysis is more complicated in this case, the dual character of the granular packing also leads to define it as dilatant if θ is less than 54.47°, and contractive if θ is greater than 54.47°.

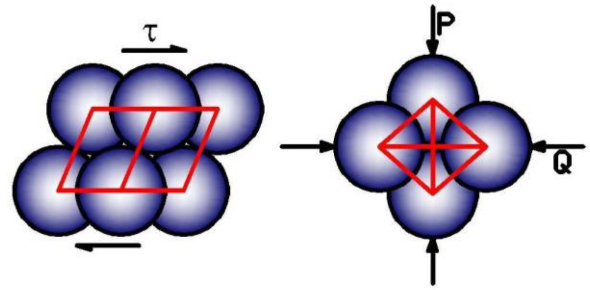


Figure 8. Mechanics of the three-dimensional stress. a) Simple shear ensemble b) Triaxial compression ensemble.

# 6 STRENGTH OF GRANULAR PACKINGS

## 6.1. Simple shear

At the time of the failure of a granular packing of acute configuration by horizontal shear, the angle of the resultant with the normal force equals the angle of internal friction, φ, and therefore: β = φ. In a granular packing of obtuse configuration, the coincidence of the centroidal reaction with the generatrix is impossible. But, since the force resultant of the top layer should be transmitted in some way to the layer below, the only rational possibility is the appearance of a shear force at the contact, which, at failure, is equal to the average shear strength between the grains, and, hence, independent of its assemblage. Then, $\beta = \varphi_{cv}$. In the first case, the angle of internal friction depends solely on the structure of the packing. Mechanically, this is only possible if the deformation is very small, and grains rotate from one another, as has been proposed by some authors (e.g. Skinner, 1969). But this mechanism is also physically impossible, unless the affected area be a narrow band that acts as a hinge between the two unaffected portions of the specimen. This is an experimental fact recognized since the beginning of soil mechanics (e.g. Taylor, 1948). Some authors argue that this band has a thickness of about ten times the diameter of the grain, based on the X-Rays analysis (e.g. Budhu, 2000). In the second case, the state of failure is reached when the grains have slipped enough respect to the neighboring grains. Consequently, the entire mass is involved, and the required deformation is relatively large. These findings are also well known from the experimental results.

The two failure mechanisms described above have a common point, in which the packing does not expand or contract itself. Indeed, Casagrande (1936) found experimentally that there was a value of the void ratio for which granular soils were strained at constant volume, and called it critical void ratio.

## 6.2. Two-dimensional compression

At failure, the following conditions hold: $\beta = \varphi$ , $\sigma_1 = \sigma_{1f}$ and $\sigma_{3i} = \sigma_{3f}$ . Then, according to equations (12) and (13),

$$\theta_f = 45^o - \frac{\varphi}{2} \quad , \qquad \sigma_{3f} = \sigma_{1f} \tan^2 \theta_f \qquad [15]$$

which coincides with the Mohr-Coulomb law for granular soils. In this equation, θf represents the plane of failure and coincides with the line of contact, as it should be. Having recognized this equivalence, all of the features deduced for the parallelogram packing are valid for the rhombic packing.

### 6.3. Triaxial compression

In this case, it is not certain that there is a simple equivalence between the prismatic force ensemble, describing the shear deformation in soils, and the rhombohedral or octahedral force ensembles, which represents the triaxial test. This is due to the effect of the second angle of assemblage α. However, a simple qualitative analysis allows for deducing that any conclusion drawn for the two-dimensional force ensemble is valid for three-dimensional force ensemble; for instance, the specimen fails along an oblique face of the polyhedron, when the soil is dense, regarding a dilatant deformation process, and it fails in a bulk fashion when the soil is loose and undergoes a contractive straining. But the derivation of the failure law for this test shall be done using another route.

## 7 RELATION BETWEEN φ AND $K_O$

The most basic but very important application of the principle of the centroidal reactions in dense soils, whose contact forces coincide with the direction of the contact lines, is the determination of the natural stress state of the subsoil. For instance, if a semi-infinite soil that extends indefinitely in depth and is limited at the top by a gently sloping surface is considered, the stress components are obtained by adding all the contact forces acting along each line of contact passing through the grain considered, which depends on the type of packing. For a three-dimensional problem, there are three or four directions, according to the packing, that can be rhombohedral or octahedral. Immediately, it shows up that the stress state is given by equation (14) for a horizontal surface; and, therefore, the coefficient of lateral thrust "at rest", $K_0$, is expressed as:

$$K_0 = \frac{\tan^2 \theta}{2} \qquad [16]$$

But when the surface is inclined, stresses vary according to the orientation of the base of the ensemble. However, whatever the orientation be, for a slope angle equal to the angle of internal friction, φ, stresses into the subsoil must meet the criterion of Mohr-Coulomb failure. Removed the angle θ from these two conditions of the surface plane, the relationship between $K_0$ and φ is achieved, which also is not unique. However, further analysis shows the values of $K_0$ bounded as follows:

$$\frac{1}{1+3\mu^2} \le K_0 \le \frac{1}{1+2\mu^2} \qquad [17]$$

where μ=tanφ is the coefficient of internal friction. Figure 9 shows that the extensive data gathered by various authors are well suited to this band, and that the lower limit approaches the empirical formula proposed by Jaky (1944).

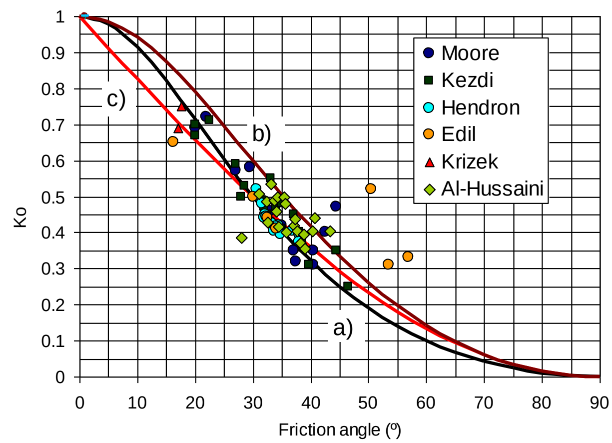


Figure 9. Relationship between the internal friction angle and the coefficient of lateral stress “at rest”. a) Lower limit, b) Upper limit, c) Jaky´s empirical relationship.

## 8 RELATIONSHIP BETWEEN φ AND POROSITY

The connection between the granular packing and the force ensemble in the triaxial compression test is the structural angle θ. If Ko is removed from equations (16) and (17), the relationship between the structural angle and the internal friction angle is obtained. Likewise, If θ is eliminated from equation (11), the following relationship is attained for the lower limit of $K_0$ :

$$\upsilon = \frac{6\chi_g}{\pi} \cos^2 \varphi \sqrt{3 - 2\cos^2 \varphi} \qquad [18]$$

This equation has been compared with the results of careful tests carried out and reported by several authors. Just for illustration, two well known data are presented in figure 10 to show their good correlation with the theoretical curves withdrawn from equation (18) for the dense state, and the horizontal line, $\varphi = \varphi_{cv}$, representing the critical friction angle, for the loose state. The intersection of this two lines gives the Casagrande´s critical void ratio.

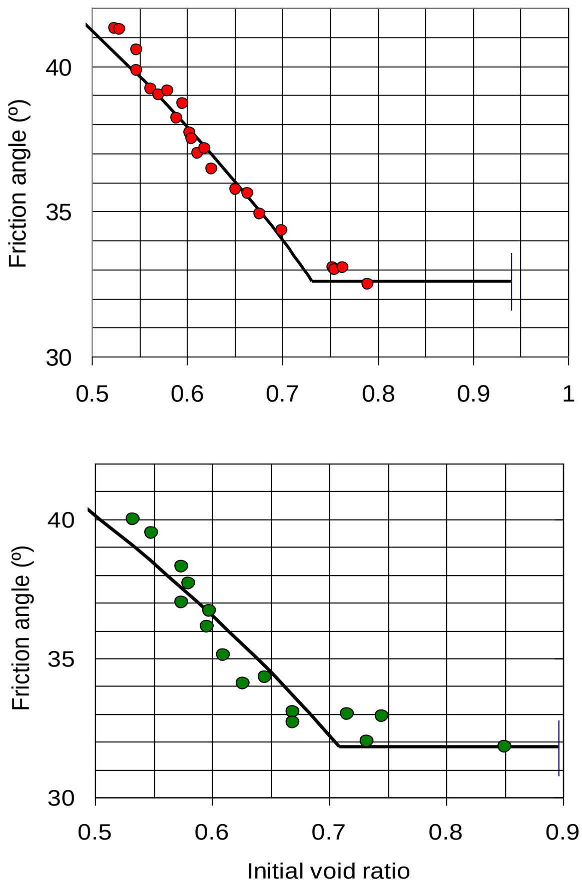


Figure 10. Relationship between the initial void ratio and the friction angle. a) for Brasted sand, $\chi_g$=1.016, (Cornforth, 1973) , b). for medium fine sand, $\chi_g$= 0.993, (Rowe,1962),

## CONCLUSIONS

To be useful for the theory of granular packings, the Wadell´s shape coefficient and the Hazen´s uniformity coefficient should be redefined. The appropriate choice of the lattice unit cell generalizes the total volume and the effective solid volume of the packing. All kinds of packings may be reduced to only three. There is a definitive relationship between the minimum and the maximum porosity in granular soils. Packings exhibit a dual character at least. Dense packings obey the principle of the centroidal reactions in studying the stress transmission under simple boundary conditions, such as soil testing. Also, the granular packing reveals in a simple manner the critical state behavior of the shear strength of soils. In general, the granular packing allows for grasping and deepening into the particulate nature of soils, and the results fit well with the experimental data. Therefore, a granular packing, so defined, is a simple, deterministic, realistic, and quantitative model for soils, so that it is the time to include it as a chapter of Soil Mechanics subject.

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