Understanding the limitations of the Swedish method of slices from the stress perspective

2011 Pan-Am CGS Geotechnical Conference

G.H. Lei, Abraham C.F. Chiu & Q. Zheng
Key Laboratory of Geomechanics and Embankment Engineering of Ministry of Education, Geotechnical Research Institute, Hohai University, Nanjing, Jiangsu, China

ABSTRACT
Swedish method of slices is widely taught in many soil mechanics courses to analyze the slope stability. From the literatures, two different equations are used to define the factor of safety by the Swedish method in the effective stress analysis. The difference lies in the effective normal stress at the base of the slices, which is derived by decomposing either the total weight or the effective weight of the slices. However, the physical assumptions underlying the two equations are not clearly explained in many soil mechanics textbooks. This paper revisits the Swedish method by conducting a stress analysis at the base of a slice, from which the assumptions and limitations of the two equations are discussed. The results show that Swedish method is only suitable to analyze the infinite slope. One of equations is valid for the uniform seepage condition and the other one is suitable for the hydrostatic condition. Besides, the computed factors of safety by both equations have greater discrepancies than those predicted by the simplified Bishop method.

RÉSUMÉ
On enseigne la méthode suédoise de tranches dans beaucoup de cours de mécanique de sol à analyser la stabilité de pente. Des literatures, deux équations différentes sont employées pour définir le facteur de la sûreté par la méthode suédoise dans l'analyse de contrainte effective. La différence se situe dans la contrainte effective normale à la base des tranches, qui est dérivée en décomposant le poids ou le poids effectif des tranches. Cependant, les prétentions physiques étant à la base des deux équations ne sont pas clairement expliquées dans les nombreux manuels de mécanique de sol. Ce article revisite la méthode suédoise en réalisant une analyse de contrainte à la base d'une tranche, de laquelle les prétentions et les limitations des deux équations sont discutées. Les résultats prouvent que la méthode suédoise est seulement appropriée pour analyser la pente infinie. Une des équations est valide pour l'état uniforme de fuite et l'autre convient à l'état hydrostatique. En outre, les facteurs de la sûreté calculés par les deux équations ont de plus grandes anomalies que ceux prévues par la méthode simplifiée de Bishop.

1 INTRODUCTION

Slope stability is one of the key components for the undergraduate soil mechanics course, among which the method of slices is taught to analyze the slope stability. In this method the soil mass above an assumed failure surface is divided by vertical planes into a series of slices. The forces acting on the slices are solved to evaluate the factor of safety (F_s) of the trial failure surface. However, the problem is statically indeterminate and many solutions have been proposed. The Swedish solution (Fellenius 1927 and 1936) is the earliest solution for the method of slices, which is still included in the syllabus of many soil mechanics courses and many design codes for practicing engineers.

When an effective stress analysis for the slope stability is conducted, there are different equations for evaluating F_s based on the Swedish solution. However, many soil mechanics textbooks do not discuss the differences between the equations, in particular their limitations in analyzing the slope stability under different seepage conditions. As the fundamental soil mechanics is the cornerstone for the students to build up the knowledge in geotechnical engineering, it is vital to clarify the issue such that the students can correctly apply the equations in different engineering applications. The objectives of the paper are first to re-examine the two common equations for the Swedish method of slices by the stress analysis using the Mohr circle. Their discrepancies for evaluating F_s are presented by comparing their computed results with other methods of analysis. Finally the limitations in analyzing the slope stability under different seepage conditions are discussed.

2 FACTORS OF SAFETY

The following two equations for F_s are commonly found in the textbooks and literatures of soil mechanics for the analysis of slope stability using the Swedish solution:

$$F_s = \frac{\sum_{i=1}^{n}\left[c'_i l_i + \left(W_i \cos\alpha_i - u_i l_i\right)\tan\varphi'_i\right]}{\sum_{i=1}^{n} W_i \sin\alpha_i} \qquad [1]$$

$$F_s = \frac{\sum_{i=1}^{n}\left[c_i' l_i + (W_i - u_i b_i)\cos\alpha_i \tan\varphi_i'\right]}{\sum_{i=1}^{n} W_i \sin\alpha_i} \quad [2]$$

where n = number of slices, c_i' = cohesion at the base of i[th] slice, φ_i' = angel of internal friction at the base of i[th] slice, α_i = inclination of the base of i[th] slice to the horizontal direction, l_i = length of the base of i[th] slice, b_i = the width of i[th] slice ($b_i = l_i \cos\alpha_i$), u_i = total pore-water pressure at the base of i[th] slice and W_i = weight of i[th] slice.

Eqn. (1) can be found in most of the soil mechanics textbooks (Whitlow 2001, Craig 2004). However Eqn. (2) has been recently suggested by a few literatures for the effective stress analysis of slope stability (Turnbull and Hvorslev 1967, Greenwood 1983 & 1985, King 1989, Morrison and Greenwood 1989, US Army Corps of Engineers 2003, Duncan and Wright 2005).

Bishop (1955) presented a calculation example to demonstrate that conservative results are obtained from Eqn. (1) in which the effective normal stress at the base of the slices may decrease to a negative value with increasing α or u. Whitman and Bailey (1967) showed that the discrepancy of the computed results between Eqn. (1) and some more rigorous methods of slices can be as high as 60%. Duncan and Wright (1980) compared the minimum factor of safety calculated by different methods of slices and showed that the differences between Eqn. (1) and some more rigorous methods of slices can be as high as 50%. Eqn. (2) was first suggested by Turnbull and Hvorslev (1967) to analyze the slope stability in the 1960s. Greenwood (1983) used the Mohr stress circle to analyze the stress at the base of the slices and derived Eqn. (2) under the assumption of zero horizontal stress. Recently some textbooks and design manuals (US Army Corps of Engineers 2003, Duncan and Wright 2005) have recommended Eqn. (2) for analyzing the slope stability.

3 STRESS ANALYSIS

The difference between Eqns. (1) and (2) is normally considered as the difference in decomposing the force components acting on the slices. In Eqn. (1) the effective normal force at the base of the slice is calculated by subtracting the total normal force at the base of the slice ($W_i \cos\alpha_i$) from the total water force at the base of the slice ($u_i l_i$). On the other hand, in Eqn. (2) the effective weight of the slice is first calculated by subtracting the total weight (W_i) from the uplift pressure ($u_i b_i$) and then it is decomposed into the effective normal force at the base of the slice ((W_i - $u_i b_i$)$\cos\alpha_i$). The corresponding effective normal stress at the base of the slice evaluated by Eqns. (1) and (2) is shown as Eqns. (3) and (4), respectively.

$$\sigma_{i1}' = \frac{W_i \cos\alpha_i}{l_i} - u_i = \sigma_{i1} - u_i \quad [3]$$

$$\sigma_{i2}' = \frac{(W_i - u_i b_i)}{l_i}\cos\alpha_i = \frac{(W_i - u_i b_i)}{b_i}\frac{b_i}{l_i}\cos\alpha_i = \sigma_{iv2}'\cos^2\alpha_i \quad [4]$$

where the subscripts 1 and 2 in Eqns. (3) and (4) represent the normal stresses are derived from Eqns. (1) and (2), respectively. In Eqn. (3), the total normal stress ($\sigma_{i1} = W_i \cos\alpha_i / l_i$) is first evaluated and then the effective stress principle is applied to calculate the effective normal stress. On the other hand, in Eqn. (4), the effective vertical stress ($\sigma_{iv2}' = (W_i - u_i b_i)/b_i$) of the slice is first evaluated and then the effective normal stress at the base of the slice is found by the Mohr circle analysis.

The Mohr circle stress method is used to analyze the stress conditions of a soil element at the base of a slice based on Eqns. (1) and (2). In the Swedish solution it is assumed that the resultant of the interslice forces is zero for each slice. Thus, the directions of the major and minor principal stresses are the vertical and horizontal directions, respectively. Figure 1a shows the total and effective stress circles at the base of a slice based on Eqn. (1). The effective normal stress (σ_{i1}') can be expressed as follows:

$$\begin{aligned}\sigma_{i1}' &= \sigma_{i1} - u_i = \sigma_v \cos^2\alpha_i - u_i \\ &= \frac{W_i}{b_i}\cos^2\alpha_i - u_i = \frac{W_i \cos\alpha_i}{l_i} - u_i\end{aligned} \quad [5]$$

It is demonstrated that Eqns. (3) and (5) are identical. Figure 1b shows the total and effective stress circles at the base of a slice based on Eqn. (2). In this case, the effective vertical stress ($\sigma_{iv2}' = \sigma_v - u_i$) is first evaluated, and then the effective normal and tangential stresses at the base of the slice are determined from the Mohr circle analysis. Eqn. (6) shows the effective normal stress derived from Eqn. (2), which is identical to Eqn. (4).

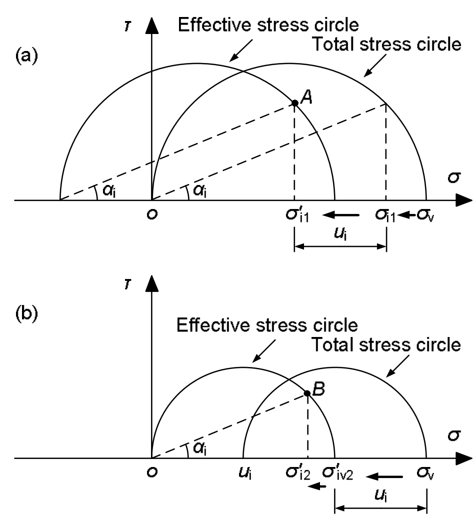

Figure 1. Stress analysis for (a) Eqn. (1), and (b) Eqn. (2)

$$\sigma'_{i2} = \sigma'_{iv2}\cos^2\alpha_i = (\sigma_v - u_i)\cos^2\alpha_i$$
$$= \left(\frac{W_i}{b_i} - u_i\right)\cos^2\alpha_i = \frac{(W_i - u_i b_i)}{l_i}\cos\alpha_i \quad [6]$$

From the above stress analysis, Eqns. (1) and (2) can lead to different effective normal stresses at the base of the slice (σ_{i1}' and σ_{i2}', respectively) because the two equations used different assumptions. Figure 1a shows that the total horizontal stress at the base of a slice is assumed zero for Eqn. (1). On the contrary, the effective horizontal stress at the base of a slice is assumed zero for Eqn. (2) in Figure 1b, in other words the total horizontal stress equals the pore water pressure. Moreover the stress conditions at the base of slice evaluated by Eqns. (1) and (2) are shown as points *A* and *B* in Figures 1a and 1b, respectively. It is demonstrated that for a given total vertical stress (σ_v), σ_{i1}' (point *A*) will become negative if α and u are sufficiently large. These limitations of Eqn. (1) are simply demonstrated with the help of the Mohr stress circle.

4 RIGID BODY ANALYSIS

The validity of Eqn. (1) can also be evaluated by the rigid body limit equilibrium method. Figure 2 shows the forces acting on a slice of soil subjected to a static ground water table. The slice can be considered as a rigid body. The total weight of the slice and the water forces at the boundary should be considered for the force equilibrium of the rigid body. In deriving Eqn. (1), no water force is considered at the vertical side of the slice. This assumption is questionable. If the pore water is incompressible, the pore water pressure should be isotropic no matter the pore water is under hydrostatic or seepage condition. The water force does not only apply at the base of the slice, but also applies on the vertical sides of the slice. Hence, a resultant water force ($u_i l_i \sin\alpha_i$) in the horizontal direction should be acted on the vertical side of the slice. Then the equilibrium equations of forces for the slice in the vertical and horizontal directions become:

$$\begin{cases} W_i = (N_i' + u_i l_i)\cos\alpha_i + T_i\sin\alpha_i \\ (N_i' + u_i l_i)\sin\alpha_i = T_i\cos\alpha_i + u_i l_i\sin\alpha_i \end{cases} \quad [7]$$

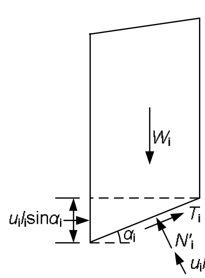

Figure 2. Force distribution of a slice

By solving Eqn. (7), the effective normal stress at the base of the slice is identical to that evaluated by Eqns. (2), (4) and (6). It should be noted that the above analysis is valid for the hydrostatic condition and the analysis of slope stability under seepage condition will be discussed later.

5 DISCREPANCIES BETWEEN TWO EQUATIONS

Regarding a homogeneous soil slope, F_s for the slope stability is influenced by the height of slope (*H*), the slope angle (β), the unit weight of soil (γ), the effective cohesion (c'), the effective internal angle of friction (φ') and pore-water pressure (*u*). Duncan and Wright (1980) recommended that the minimum factor of safety may be used to evaluate the accuracy of different methods of analysis. To take into account the effect of pore-water pressure on the stability of slope, the pore-water pressure ratio $r_u = u_i/(\gamma h_i)$ (Bishop and Morgenstern 1960) is used to describe the distribution of the pore-water pressure along the slip surface. In order to minimize the effects of different combinations of parameters on the results of calculations, F_s evaluated by the limit analysis using a log spiral sliding mechanism is considered as the reference for the other methods of analysis. The following five methods are evaluated by Duncan and Wright (1980): Swedish, simplified Bishop, Janbu, Spencer, Morgenstern and Price. For a given β and r_u, $\lambda = \gamma H\tan\varphi'/c'$ and $F_s/\tan\varphi'$ can exhibit a unique relationship based on the limit analysis. Thus, for any combinations of *H*, γ, c' and φ', F_s is unique provided that β, r_u and λ are given. They have shown that the errors of the Swedish method based on Eqn. (1) increases with increasing r_u and λ. When $r_u = 0.6$ and $\lambda = 50$, the errors can reach 50%. However, the maximum error from the other methods of slices (including the simplified Bishop method) does not exceed 10%.

Despite previous studies (Turnbull and Hvorslev 1967, Greenwood 1983, 1985, U.S. Army Corps of Engineers 2003, Duncan and Wright 2005) have recommended Eqn. (2) when the Swedish method of slices is used for the slope stability analysis, the accuracy of Eqn. (2) has not well studied. Thus the following example is used to compare the calculating errors between Eqns. (1) and (2). The comparison is based on the methodology presented in Duncan and Wright (1980). As a reference point, the limit analysis based on a logarithmic spiral sliding mechanism is performed for the stability of a slope (Michalowski 2002). The relationship between $c'/(\gamma H\tan\varphi')$ and $F_s/\tan\varphi'$ is derived from the limit analysis (see Fig. 3 in Michalowski 2002). The following parameters are used for the calculations: $F_s = 1$, $r_u = 0.5$, $\gamma = 20$ kN/m^3, $H = 10$ m, $\beta = 15^o, 30^o, 45^o, 60^o, 75^o$, λ = 2, 3, 4, 5, 10, 20 and 50. The analysis was carried out using the computer program SLP (Yin et al. 1992). In the original SLP program, the term $W_i\cos\alpha_i - u_i l_i$ found in Eqn. (1) is assumed greater than or equal to zero. This methodology is denoted as Eqn. (1*) in this paper. The SLP program was modified in this study. The modifications are (i) the pore-pressure coefficient r_u is used to evaluate the pore-water pressure, (ii) Eqns. (1), (2) and simplified Bishop method are implemented to calculate the minimum factor of safety. The computation parameters and calculation results are shown in Table 1.

Consider F_s = 1 (obtained from the limit analysis) is the reference point, Table 1 shows that among the four methods, the results predicted by the simplified Bishop method are the nearest to those predicted by the limit analysis (i.e. F_s = 1). The maximum discrepancy is below 6%. For the other three methods based on the Swedish method of slice, the discrepancy between Eqn. (1) and the limit analysis is generally the greatest. For $\beta = 45^o$, F_s decreases with increasing λ. For $\beta = 45^o$ and $\lambda = 50$, F_s = 0.424, the discrepancy is greater than 50%. Further, for $\beta = 60^o$ and $\lambda = 20$, F_s = 0.125, the discrepancy is greater than 80%. However, F_s predicted by Eqn. (2) is in general greater than 1, i.e. not conservative. It is observed F_s increases with increasing λ. For $\beta = 45^o$ and $\lambda = 50$, F_s = 1.813, the discrepancy is greater than 80%. Further, for $\beta = 60^o$ and $\lambda = 20$, F_s = 2.647, the discrepancy is greater than 160%. In Eqn. (1*), the term $W_i\cos\alpha_i - u_il_i$ in Eqn. (1) is assumed greater than or equal to zero. As a result, the discrepancy between Eqn. (1*) and limit analysis is reduced. It is observed, F_s increases with increasing β, For $\beta = 60^o$, F_s increases with increasing λ, at $\lambda = 20$, F_s = 1.39, the discrepancy is about 39%. Further, for $\beta = 75^o$ and $\lambda = 5$, F_s = 1.493, the discrepancy is about 50%. Thus, based on the results presented in Table 1, no matter using Eqns. (1), (2) and (1*), the results predicted by the Swedish method have greater discrepancies than those predicted by the simplified Bishop method, in particular for slope with higher angle.

6 SLOPE STABILITY UNDER SEEPAGE CONDITIONS

A few previous studies (King 1989, Morrison and Greenwood 1989) have pointed out that Eqn. (1) is only applicable to evaluate the stability of an infinite slope subjected to uniform seepage condition (i.e. the flow direction is parallel to the slope angle) and Eqn. (2) is only applicable to hydrostatic condition. Thus, both equations may not be applied directly to a slope under a general seepage condition. King (1989) used the forces in the slice and the corresponding vectors diagram to illustrate his idea. On the other hand, Morrison and Greenwood (1989) used the water pressure at the interslice to analyze the problem. Further discussion of these approaches have been addressed by Sarma et al. (1990) and Chugh and Li (1990). According to the stress analysis presented in this study, the total horizontal stress of the slice is assumed zero for Eqn. (1) which is only satisfied by the infinite slope subjected to uniform seepage. On the other hand, the effective horizontal stress of the slice is assumed zero for Eqn. (2) which is only satisfied by the infinite slope subjected to hydrostatic condition.

Table 1. Comparison between factors of safety calculated by the Swedish method of slices (Eqns. (1), (2) and (1*) and simplified Bishop method)

β (^{o})		15	15	15	15	30	30	30	30	45	45
λ		5	10	20	50	5	10	20	50	5	10
$F_s/\tan\varphi'$ (a)		4.050	3.130	2.580	2.190	2.535	1.763	1.310	0.980	1.817	1.127
$\tan\varphi'$ (b)		0.247	0.319	0.388	0.457	0.395	0.567	0.763	1.020	0.550	0.887
φ' (^{o}) (b)		13.9	17.7	21.2	24.5	21.5	29.6	37.4	45.6	28.8	41.6
c' (kPa) (c)		9.9	6.4	3.9	1.8	15.8	11.3	7.6	4.1	22.0	17.7
F_s	Eqn. (1)	0.808	0.811	0.829	0.866	0.842	0.787	0.757	0.760	0.851	0.771
	Eqn. (2)	0.956	0.970	0.919	1.017	1.036	1.063	1.116	1.193	1.123	1.251
	Eqn. (1*)	0.851	0.846	0.862	0.888	0.904	0.879	0.870	0.866	0.948	0.953
	Simplified Bishop	0.995	0.946	0.998	0.995	0.998	0.994	0.993	0.988	0.973	0.986
β (^{o})		45	45	60	60	60	60	75	75	75	75
λ		20	50	2	5	10	20	2	3	4	5
$F_s/\tan\varphi'$ (a)		0.740	0.450	2.989	1.280	0.650	0.310	2.295	1.454	0.991	0.707
$\tan\varphi'$ (b)		1.350	2.222	0.335	0.781	1.539	3.226	0.436	0.688	1.009	1.414
φ' (^{o}) (b)		53.5	65.8	18.5	38.0	57.0	72.8	23.5	34.5	45.3	54.7
c' (kPa) (c)		13.5	8.9	33.5	31.3	30.8	32.3	43.6	45.9	50.5	56.6
F_s	Eqn. (1)	0.634	0.424	0.981	0.919	0.742	0.125	1.029	1.037	1.048	1.033

	Eqn. (2)	1.404	1.813	1.154	1.384	1.793	2.647	1.267	1.457	1.73	2.076
	Eqn. (1*)	0.931	0.949	1.037	1.094	1.195	1.390	1.097	1.181	1.319	1.493
	Simplified Bishop	0.969	0.968	0.973	0.985	0.982	0.947	0.953	0.962	0.976	0.983

(a) $F_s/\tan\varphi'$ is evaluated from Fig. 3 in Michalowski (2002)
(b) $\tan\varphi'$ and φ' are calculated from $F_s/\tan\varphi'$ assuming $F_s = 1$
(c) c' is calculated from H, λ, γ and $\tan\varphi'$

7 CONCLUSIONS

Two common equations for the factor of safety based on the Swedish method of slices are re-examined by the stress analysis and rigid body limit equilibrium analysis. Furthermore a calculating example is used to compare the accuracies of the two equations with the simplified Bishop method. The following conclusions can be drawn:

(i) The underlying assumptions of Eqns. (1) and (2) are different. Eqn. (1) assumes the total horizontal stress at the base of a slice is zero, but Eqn. (2) assumes the effective horizontal stress at the base of a slice is zero.

(ii) The limitations of Eqns. (1) and (2) are identified. Eqn. (1) is valid for the analysis of infinite slope subjected to uniform seepage and Eqn. (2) is suitable to analyze the infinite slope under hydrostatic condition.

(iii) The factors of safety computed by Eqns. (1) and (2) have greater discrepancies than those predicted by the simplified Bishop method, in particular for slope with higher angle.

ACKNOWLEDGEMENTS

The writers would like to acknowledge the financial supports from the Qing Lan Project of Jiangsu Province of China, from the Fundamental Research Funds for the Central Universities of China and from the research grants funded by the National Natural Science Foundation of China under grant nos. 50878075 and 50878076.

REFERENCES

Bishop, A.W. 1955. The use of the slip circle in the stability analysis of slopes, *Géotechnique*, 5: 7-17.
Bishop, A.W. and Morgenstern, N. 1960. Stability coefficients for earth slopes, *Géotechnique*, 10: 129-150.
Chugh, A.K. and Li, K.S. 1990. Discussion of 'Assumptions in simplified slope stability analysis by the method of slices', *Géotechnique*, 40: 655-658.
Craig, R.F. 2004. *Soil mechanics*, 7th ed., Spon Press.
Duncan, J.M. and Wright, S.G. 1980. The accuracy of equilibrium methods of slope stability analysis, *Engineering Geology*, 16: 5-17.
Duncan, J.M. and Wright, S.G. 2005. *Soil strength and slope stability*, John Wiley & Sons, New York, NY, USA.
Fellenius, W. 1927. *Erdstatische Berechnungen mit Reibung und Kohäsion (adhäsion) und unter Annahme kreiszylindrischer Gleitflächen*, Verlag Ernst & Sohn, Berlin.
Fellenius, W. 1936. Calculation of the stability of earth dams, *2nd Congress on Large Dams*, Washington, D.C., USA, 4: 445-462. (reprinted in Marr, W.A. 2003. A history of progress: selected U.S. papers in geotechnical engineering. *Geotechnical Special Publication No. 118, ASCE*, 1: 129-146).
Greenwood, J.R. 1983. A simple approach to slope stability, *Ground Engineering*, 16: 45-48.
Greenwood, J.R. 1985. Discussion of 'stability of compacted rockfill slopes', *Géotechnique*, 35: 217-218.
King, G.J.W. 1989. Revision of effective-stress method of slices, *Géotechnique*, 39: 497-502.
Michalowski, R.L. 2002. Stability charts for uniform slopes, *Journal of Geotechnical and Geoenvironmental Engineering*, ASCE, 128: 351-355.
Morrison, I.M. and Greenwood, J.R. 1989. Assumptions in simplified slope stability analysis by the method of slices, *Géotechnique*, 39: 503-509.
Sarma, S.K., Morrison, I.M. and Greenwood, J.R. 1990. Discussion of 'Revision of effective-stress method of slices', *Géotechnique*, 40: 651-654.
Turnbull, W.J. and Hvorslev, M.J. 1967. Special problems in slope stability. *Journal of the Soil Mechanics and Foundations Division*, ASCE, 93: 499-528.
U.S. Army Corps of Engineers. 2003. *Engineering and design: slope stability*, Engineer Manual No. EM1110-2-1902, Department of the Army, Washington, DC, USA.
Whitlow, R. 2001. *Basic soil mechanics*, 4th ed., Prentice Hall.
Whitman, R.V. and Bailey, W.A. 1967. Use of computers for slope stability analysis. *Journal of the Soil Mechanics and Foundations Division*, ASCE, 93: 475-498.
Yin, Z.Z., Guo, Z.P., Xu, H.J., ZHU, Z.M., Wang, X.Z. and Yan, Z.H. 1992. A program for slope stability analysis (SLP), In *Modern computation methods and programs of hydraulic engineering and geotechnical engineering*, 343-351, Hohai University Press, Nanjing, China. (in Chinese)